Redundancy and Multiple Objectives in Linear Robust Control

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To my grandparents,

in gratitude for your love, leadership and humour.

Dear Lord God,

you know this human frame and have suffered in love to restore it.
May my life give praise to you.

In the Letter to the Hebrews, Chapter 4, verses 14–16:
Since, then, we have a great high priest who has passed through the heavens, Jesus, the Son of God, let us hold fast to our confession. For we do not have a high priest who is unable to sympathize with our weaknesses, but we have one who in every respect has been tested as we are, yet without sin. Let us therefore approach the throne of grace with boldness, so that we may receive mercy and find grace to help in time of need.

Statement of Originality

These doctoral studies were conducted under the supervision of Professor Brian Anderson and with Dr. Michael Green and Dr. Duncan MacFarlane as advisors.

The work presented in this thesis is the result of original research carried out by myself in cooperation with Professor Brian Anderson and a number of researchers from other universities; Professor Tsutomu Mita (Chiba University, Japan), Dr David Clements (University of New South Wales), Professor Alan Laub (University of California, Santa Barbara) and Dr David Limebeer (Imperial College, London). All work was carried out while enrolled in the Department of Systems Engineering as a Doctor of Philosophy student. Approximately 70% of the work presented is my own. The research reported on in this thesis has not been submitted for any other degree or award in any other university or educational institution.

JBMatson

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August 1995

Acknowledgements

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Many thanks also to the academic and administrative staff in the Department of Systems Engineering for their comradeship, encouragement and helpfulness during my stay in the department. I have been fortunate to share the experience of doing a PhD with other students in the Department; in particular, for their friendship and many discussions both technical and otherwise I would like to thank Wee Sit Lee, Wee Sun Lee, Ari Partanen, Anton Madievski, Craig Watkins, Kim and Perry (PERRY) Blackmore and Mehmet Karan.

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It is a privilege to have been able to spend the past three and a half years engaged in full time research work. I have many to thank for this opportunity, not least of these is the Australian taxpayer who has supported me financially and the Australian National University where I have been given the opportunity to study.

To comrades in moments of trial, pain and much joy, Aidan (Seamus) Cahill, Ken('nuth) Yamazaki, Richard (Tricky) Walker, Derek (Degsy) Brookes and Dave Campbell, I thank you for your patience, prayer and careful listening. Many thanks to other dear friends far and wide whose encouragement, love and acceptance is far beyond mere words, thanks for your comradeship and support on this journey.

My family, you are so precious a gift.
For all you have given me, my love and thanks.
For helping me to complete this goal, yet also to look to the future in faith and hope.

Doing a PhD is a life experience, only one part of which has been writing this document. In submitting it, the author takes heart in the following piece of scripture:

In Paul's letter to the Romans, Chapter 8, verse 28:
We know that all things work together for good for those who love God, who are called according to His purpose.

Preface

That engineers are involved in the realization of technical goals which are of significance in a broader social context is implicit in the first of the tenets in the code of ethics of the *Institution of Engineers Australia:*

Members shall at all times place their responsibility for the welfare, health and safety of the community before their responsibility to sectional or private interests or to other members.

The outcomes of an engineer's work are more than just the provision of industrial equipment, consumer goods, public works, to name but a few immediately tangible examples. These outcomes generally constitute the means by which broader economic, social, industrial and cultural goals are attained.

The fact that all engineering occurs in such a context should give the engineer cause to reflect upon how he interacts with society and how he chooses to organize his activities in response. Traditionally the focus of engineering has been on the realization of goals at a technical or scientific level, rather than on taking an active role in the definition of these goals. Engineers have generally not been trained to reflect on the broader social dimensions of their vocation, although these have always been present. Increasingly, both of these aspects must be viewed together. The demand of the public on engineers is for technological solutions which are of the highest quality, are safe, environmentally sustainable, efficient, economical and which are user-friendly. To provide these solutions, the technology development process must adhere to the highest technical standards and yet simultaneously be driven by a discourse with the end user.

A serious consideration of these issues also involves questions of political involvement, social responsibility and personal moral decisions. These issues are of concern to the profession, as evidenced by the recent debate concerning professional ethics and codes of regulation. Such questions are also being raised anew in other professions. Whilst the espousement of codes of ethics is essential, such codes cannot in themselves create ethical behaviour. This endeavour might be aided by the application of systematic approaches to dealing with such issues as part of the engineer's everyday duties. Environmental impact assessments are one example of such a strategy. The challenge is to apply intellectual and other resources in managing the society-technology interface as an integral part of every subfield of engineering. This is bound to lead to increased complexity in engineering design tasks and further challenging work for researchers.

Complex decisions must often be made with limited resources and information. The engineer must prioritize his decisions and assess the relative benefits of different actions. It is basic that the engineer must acknowledge the limit to his knowledge during the design process. It is commonsense that basing decisions solely on a 'best guess' is an unwise practice. Fortunately, it is often possible to describe the bounds of one's uncertainty. This, in itself, provides valuable information for decision making. This is a principle which finds expression in the field of *robust control design* to which this thesis makes a contribution. The work presented is offered in good faith, in the hope that it will be of good use.

Glossary

ARE	algebraic Riccati equation.
DARE	discrete algebraic Riccati equation.
FI	full information.
KF	Kalman filter.
LQG	Linear-quadratic-Gaussian.
OE	output estimation.
RDE	Riccati difference equation.

Some notes on English usage.

- The personal pronoun *he* is used to refer to a person, whether they be male or female.
- After much agonizing, the letter *z* has been chosen over the letter *s* where such a choice is possible.

Abstract

Control Design Perspective.

This thesis reports on investigations related to two technical questions which have relevance to linear robust control design with multiple synthesis objectives. The results obtained in each case also have relevance to design with a single synthesis objective as well as to other systems and control problems. The unifying goal for these investigations has been to find easily implementable algorithms to aid the controller synthesis part of the control design process. Some attention is first given to the design process as a whole and the role of controller synthesis before specific technical questions are addressed. A simple design example is introduced which makes this background material concrete. The posing of a controller synthesis problem, quite aside from its solution, is a major step in the design process. The perspective taken is that controller synthesis theory should support this process as much as possible. Standard software synthesis tools generally stipulate that the controller synthesis problem satisfy certain mathematical assumptions, which in effect impose constraints on the designer. It is desirable that these assumptions put as few additional constraints as possible on the creative activity of design. The algorithms presented here do not in themselves, nor seen together, constitute full synthesis procedures for multiple objective robust control design. Rather, they are intended to provide tools in the development of such procedures, an activity which remains a challenging research topic.

Redundancy due to additional Sensors and Actuators in $\mathcal{H}_\infty$ Control.

Necessary and sufficient conditions for existence, and full parametrizations are derived for $\mathcal{H}_\infty$ controllers of a class of state-space realizations of linear, time-invariant generalized plants which is somewhat broader than the class which is considered in the so-called *standard* $\mathcal{H}_\infty$ synthesis theory (see e.g. [25]). Assumptions from the standard theory concerning the dimensions of the generalized plant's disturbance input and objective output signal spaces are relaxed. This allows for the possibility that the generalized plant has more control inputs than $\mathcal{H}_\infty$ objective signals and/or more measurements than disturbances associated with the $\mathcal{H}_\infty$ objective. In such cases there is some *redundancy* in the control inputs and/or in the measurements. Controller synthesis problems with redundancy are likely to arise in design scenarios where the $\mathcal{H}_\infty$ objective is just one of many synthesis objectives associated with different input/output signal pairs of the same generalized plant. A design example with a mixed $\mathcal{H}_2$ / $\mathcal{H}_\infty$ synthesis objective is intro-

duced to illustrate this point. The nonstandard $\mathcal{H}_\infty$ controller parametrizations contain free stable transfer function matrix parameters which are not present in the standard $\mathcal{H}_\infty$ controller parametrization. These additional parameters make the redundancy in control laws explicit. Parametrizations of nonstandard $\mathcal{H}_2$ control laws can be derived using similar techniques to those which are used to derive the nonstandard $\mathcal{H}_\infty$ results. A summary of the nonstandard $\mathcal{H}_2$ results is presented without proof.

Spectral Factorization with Imaginary Axis and Unit Circle Zeros.

Controller synthesis algorithms with an $\mathcal{H}_\infty$ or an $\mathcal{H}_2$ objective are widely accessible in standard software packages. However, synthesis algorithms which enable $\mathcal{H}_\infty$ and $\mathcal{H}_2$ objectives to be *simultaneously* achieved on possibly different input/output pairs for the same generalized plant are not. This work investigates one aspect of a mixed $\mathcal{H}_2$ / $\mathcal{H}_\infty$ synthesis objective where optimal $\mathcal{H}_2$ performance on one input/output signal pair of the generalized plant is sought, subject to the satisfaction of an $\mathcal{H}_\infty$ bound on another, in general different, input/output signal pair. Both the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ synthesis objectives have connections with factorization theory for rational spectral matrices. In particular, the satisfaction of a bound on the $\mathcal{H}_\infty$ norm of a closed loop transfer function matrix is equivalent to nonnegative definiteness of a certain spectral matrix on the imaginary axis. The bounded real lemma says that this condition is equivalent to the existence of a strong solution of a certain algebraic Riccati equation. It is shown that the the bound on the $\mathcal{H}_\infty$ norm of the closed loop must generally be *achieved* by optimal mixed $\mathcal{H}_2$ / $\mathcal{H}_\infty$ control laws. In this case, the spectral matrix associated with the $\mathcal{H}_\infty$ bound has a transmission zero on the imaginary axis. Algorithms in standard software for solving the algebraic Riccati equation fail in such circumstances due to the fact that these algorithms preclude realizations of spectral matrices which have imaginary axis invariant zeros (whether they be transmission or decoupling zeros).

An algorithm is developed for solving the algebraic Riccati equation whose implementation is straightforward and which has known convergence properties when the spectral matrix realization has imaginary axis invariant zeros. The proposed algorithm relies on bilinear transformation of the continuous time spectral matrix to form a discrete time spectral matrix. The resulting realization of the discrete time spectral matrix has unit circle invariant zeros whenever the realization of the original continuous time spectral matrix has imaginary axis invariant zeros. The algebraic Riccati equation associated with the continuous time spectral matrix has the same strong solution as the discrete algebraic Riccati equation associated with the discrete time spectral matrix. An algorithm for solving the discrete ARE is proposed which relies on the convergence of a related Riccati difference equation. It is shown that the Riccati difference equation converges at a known rate ($\frac{1}{k}$) to the strong solution of the ARE. Secondly, it is shown how a doubling algorithm can be used to calculates RDE iterates only at integral powers of two. The result is an algorithm with known convergence rate which can be used to solve a wide range of continuous or discrete spectral factorization problems.

Contents

Notation, Definitions and Fundamental Results

General.

- Given any $z \in \mathbb{C}$, let $\Re\{z\}$ and $\Im\{z\}$ denote its real and imaginary parts, respectively.
- Given any two nonnegative integers j and k, define
$$\binom{j}{k} = \begin{cases} 0 & \text{if } j < k \\ \frac{j!}{(j-k)!k!} & \text{otherwise} \end{cases} . \tag{0.1}$$
- Define the discrete time unit step function $u(\cdot)$ as follows; with l any integer, $u(l) = 1$ if $l \geq 0$ and $u(l) = 0$ if $l < 0$.
- Given $f(l)$ and $g(l)$, both scalar functions of an integer variable l, we say $g(l) = \mathcal{O}(f(l))$ if there exists a constant $0 \leq \kappa < \infty$ such that $\lim_{l\to\infty} \frac{|g(l)|}{|f(l)|} = \kappa$.

Linear algebra.

- Let 0_m denote the $m \times m$ zero matrix, $0_{n\times m}$ the $n \times m$ zero matrix and I_m the $m \times m$ identity matrix.
- Given a matrix $M \in \mathbb{C}^{n\times m}$, M^T denotes its transpose and M^* its complex conjugate transpose.
- Given a square matrix $M \in \mathbb{C}^{n\times n}$, $\{\lambda_i(M)\}$ $i = 1, \ldots n$ denotes the set of eigenvalues of M and $\rho(M) = \max_i |\lambda_i(M)|$ the spectral radius of M.
- Given a matrix $M \in \mathbb{C}^{n\times m}$, let $\{\sigma_j(M)\}$ $j = 1, \ldots, \min\{n, m\}$ denote the set of singular values of M and $\bar{\sigma}(M) = \sigma_{\max}(M)$ the maximum singular value of M.
- A matrix $M \in \mathbb{R}^{n\times m}$ is full rank if $\text{rank}\{M\} = \min\{n, m\}$.
- Given a matrix $M \in \mathbb{R}^{n\times n}$, we say that any p-dimensional linear subspace $\mathcal{W} \subset \mathbb{R}^n$ is M-invariant if $M\mathcal{W} \subset \mathcal{W}$. If $V \in \mathbb{R}^{n\times p}$ is a full column rank matrix whose columns span $\mathcal{W}$, then there exists a matrix $M_V \in \mathbb{R}^{p\times p}$ such that $MV = VM_V$.
- Given a matrix M, we let M_{ij} or $(M)_{ij}$ denote the $(i, j)^{\text{th}}$ entry of M.
Suppose M has an even number of rows and columns, consisting of a matrix of 2×2 matrix sub-blocks; for convenience we let $[M]_{ij} \in \mathbb{R}^{2\times 2}$ denote the $(i, j)^{\text{th}}$ 2×2 subblock of M.

- Let $f(l)$ be a scalar valued function and $U(l)$ be a matrix valued function, both of an integer variable l. We say that $U(l) = \mathcal{O}(f(l))$ if $\sigma_{\max}(U(l)) = \mathcal{O}(f(l))$, where the notation $\mathcal{O}(\cdot)$ applied to scalar functions has the definition given above.
  Note the following property: If $U(l)$ is such that each $(U(l))_{mn} = \mathcal{O}(f(l))$ or each $[U(l)]_{ij} = \mathcal{O}(f(l))$, then $U(l) = \mathcal{O}(f(l))$.

The matrix inversion lemma.

Suppose one is given the matrices A, B, C, D, with both A and C invertible.
The matrix $(A+BCD)$ is invertible if and only if the matrix $(DA^{-1}B+C^{-1})$ is invertible.
Moreover, the following identity holds:

$$(A+BCD)^{-1} = A^{-1} - A^{-1}B(DA^{-1}B+C^{-1})^{-1}DA^{-1}. \tag{0.2}$$

Signals.

- $\mathcal{L}_2(-\infty,\infty)$ denotes the set of square-integrable real vector-valued signals; $l(\cdot) \mid \mathbb{R} \rightarrow \mathbb{R}^n$.
  Given any $l(\cdot) \in \mathcal{L}_2(-\infty,\infty)$, define the 2-norm of $l(\cdot)$ as $\|l\|_2 = \left\{\int_{-\infty}^{\infty} l^T(t)l(t)dt\right\}^{\frac{1}{2}}$.

- $\mathcal{L}_2$ denotes the set of vector-valued frequency domain signals, $l(\cdot) \mid \mathbb{C} \rightarrow \mathbb{C}^n$ with $l(j\omega)^* = l^T(-j\omega)$, which are square-integrable on the imaginary axis.
  Given any $l(\cdot) \in \mathcal{L}_2$, define the 2-norm of $l(\cdot)$ as $\|l\|_2 = \left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} l^T(-j\omega)l(j\omega)d\omega\right\}^{\frac{1}{2}}$.

- $\mathcal{L}_2$ and $\mathcal{L}_2(-\infty,\infty)$ are Hilbert spaces, isomorphic under the Fourier transform

$$\mathcal{F}\{l\} = \int_{-\infty}^{\infty} l(t)e^{-j\omega t}dt. \tag{0.3}$$

- $\mathcal{L}_2[0,\infty)$ denotes the subspace of signals in $\mathcal{L}_2(-\infty,\infty)$ which are zero for $t \in (-\infty, 0)$.
  $\mathcal{L}_2[0,\infty)$ inherits the norm $\|\cdot\|_2$ from $\mathcal{L}_2(-\infty,\infty)$.

- $\mathcal{H}_2$ is the set of complex vector valued signals $l(\cdot) \mid \mathbb{C} \rightarrow \mathbb{C}^n$ which are analytic in the open right half plane and for which the following norm exists:

$$\|l\|_2 = \left\{\sup_{\alpha>0} \frac{1}{2\pi}\int_{-\infty}^{\infty} l(\alpha+j\omega)^* l(\alpha+j\omega)d\omega\right\}^{\frac{1}{2}}. \tag{0.4}$$

- The spaces $\mathcal{L}_2[0,\infty)$ and $\mathcal{H}_2$ are isomorphic under the Laplace transform.
  Given a signal $l(\cdot) \mid \mathbb{R} \rightarrow \mathbb{R}^n$, its Laplace transform, when it exists, is defined as follows:

$$\hat{l}(s) = \mathcal{L}\{l(t)\} = \int_0^{\infty} l(t)e^{-st}dt. \tag{0.5}$$

When the context ensures no ambiguity, $l(s)$ ($s \in \mathbb{C}$) will be used to denote the Laplace domain representation $\hat{l}(s)$ of a time domain signal $l(t)$ ($t \in \mathbb{R}$).

- Let $y(t) \in \mathbb{R}^n$ denote a vector-valued continuous-time stationary stochastic process.
  $\mathcal{E}\{y(t)\}$ denotes the expectation of $y(t)$.
  $\phi_{yy}(\tau) = \mathcal{E}\{y(t)y^T(t+\tau)\}$ is the autocorrelation function of $y(t)$.
  $\Phi_{yy}(j\omega) = \mathcal{F}\{\phi_{yy}(\tau)\}$ is the power spectral matrix of $y(t)$, where $\mathcal{F}\{\cdot\}$ denotes the Fourier transform.

Finite dimensional linear time invariant (FDLTI) systems.

The behaviour of a FDLTI system (operator), mapping continuous time input signals $u(t) \in \mathbb{R}^n$ to output signals $y(t) \in \mathbb{R}^m$ with $t \in (0, \infty)$, may be described by its impulse response matrix $M(t)$ as follows:

$$y(t) = \int_0^\infty M(t-\tau)u(\tau)d\tau. \tag{0.6}$$

A FDLTI system can also be expressed in terms of a real-rational transfer function matrix $\hat{M}(s) \mid \mathbb{C} \rightarrow \mathbb{C}^{m \times n}$ via the Laplace transform of its impulse response matrix $M(t)$:

$$\hat{M}(s) = \mathcal{L}\{M(t)\} = \int_0^\infty M(t)e^{-st}dt, \tag{0.7}$$

in which case $\hat{y}(s) = \hat{M}(s)\hat{u}(s)$, provided all initial conditions are zero.
When the context ensures no ambiguity, $M(s)$ will be used to denote $\hat{M}(s)$.
The Laplace variable s is often omitted for brevity; i.e. $M = M(s)$.

- Let $\mathcal{R}$ denote the set of matrix-valued real-rational functions of a complex variable.
- normrank$\{M(s)\}$ denotes the *normal rank* of $M(s) \in \mathcal{R}$ which is its rank over the set of scalar real-rational functions of $s \in \mathbb{C}$.
  If normrank$\{M(s)\} = m$ then rank$\{M(s)\} = m$ for all $s \in \mathbb{C}$, except at a finite number of points.
- $M(s)$ is *proper* if $\lim_{s \to \infty} M(s)$ is finite.
  All FDLTI systems introduced will be assumed proper, unless otherwise stated.
- We call $M(s)$ *strictly proper* if $\lim_{s \to \infty} M(s) = 0$.
- $M(s) \in \mathcal{H}_\infty$ if $M(s)$ is analytic in $\Re\{s\} > 0$ and the infinity norm of $M(s)$ is well defined;

$$\|M(s)\|_\infty = \sup_{\alpha > 0} \left\{ \sup_{\omega} \{\bar{\sigma}(M(\alpha + j\omega))\} \right\} < \infty. \tag{0.8}$$

- $M(s) \in \mathcal{RH}_\infty$ if $M(s)$ is real-rational and in $\mathcal{H}_\infty$.
  $M(s) \in \mathcal{RH}_\infty$ if and only if it is real-rational, proper and analytic in $\Re\{s\} \geq 0$.

- If $M(s) \in \mathcal{RH}_\infty$, then $\|M\|_\infty$ is the supremum of $\bar{\sigma}(M(j\omega))$ over $\omega \in [-\infty, \infty]$. $\|\cdot\|_\infty$ is *submultiplicative*; i.e. if $M, N \in \mathcal{RH}_\infty$ then $\|MN\|_\infty \leq \|M\|_\infty \|N\|_\infty$.

- $M(s) \in \mathcal{RA}^+$ if $M(s)$ is real-rational and analytic in $\Re\{s\} > 0$. Note that $\mathcal{RH}_\infty \subset \mathcal{RA}^+$, but the reverse is not true since $\mathcal{RA}^+$ includes systems for which the infinity norm is not well defined, e.g. those with imaginary axis poles.

- $M(s) \in \mathcal{BH}_\infty^\gamma$ if $M(s) \in \mathcal{RH}_\infty$ and $\|M(s)\|_\infty < \gamma$. $M(s) \in \overline{\mathcal{BH}_\infty^\gamma}$ if $M(s) \in \mathcal{RH}_\infty$ and $\|M(s)\|_\infty \leq \gamma$.

- If needed, superscripts may be added to any of the above sets of FDLTI systems to indicate the dimension of the input and output spaces.

Well Posedness and Internal Stability of Feedback Connections.

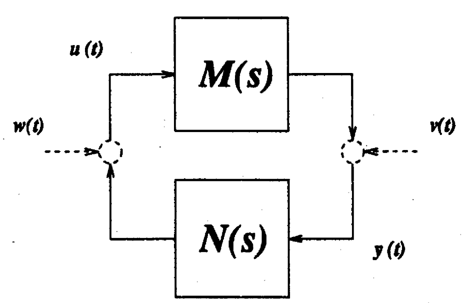

Figure 0.1: Internal Stability of a feedback loop.

A feedback loop consisting of two FDLTI systems as depicted in Figure 0.1 is said to be *well-posed* if the inverse $(I - M(\infty)N(\infty))^{-1}$ exists. This ensures the existence and properness of the inverse $(I - M(s)N(s))^{-1}$.

The feedback loop in Figure 0.1 is said to be *internally stable* if the response of the internal feedback signals $u(t)$ and $y(t)$ to the fictional perturbation signals $w(t)$ and $v(t)$ is stable. In broad terms, a small perturbation at any point in the feedback loop will result in a variation from nominal behaviour elsewhere in the loop which is also small.

The feedback loop in Figure 0.1 is internally stable if and only if

$$\begin{pmatrix} I & -K \\ -G & I \end{pmatrix} \in \mathcal{RH}_\infty. \tag{0.9}$$

See Lemma A.1.1 for a proof and discussion of this result. For a discussion of internal stability, see e.g. [34] or [33].

Linear Fractional Maps.

Suppose one is given a FDLTI system $M(s) \in \mathcal{R}$ which has two vector-valued input signals u_1 and u_2 and two vector-valued output signals y_1 and y_2. Suppose one is given another system $N(s)$ which accepts input signals with the same dimension as y_2 and

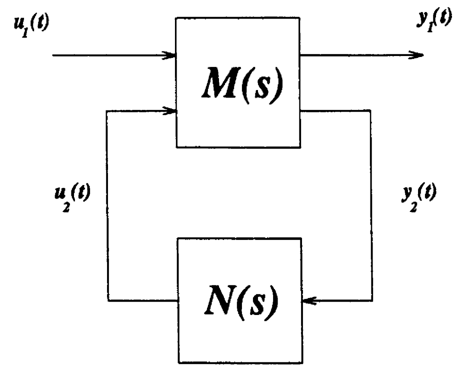

Figure 0.2: Feedback configuration for the linear fractional map notation.

has outputs with the same dimension as u_2. Let the two systems be connected in the feedback configuration $u_2 = N(s)y_2$ as depicted in Figure 0.2. Let $M(s)$ have block partitions $M_{ij}(s)$ $i,j \in \{1,2\}$ whose dimensions correspond with those of the input and output signal vectors. The resulting closed loop system has input/output behaviour which can be described by the *linear fractional map* $y_1 = LFT\{M,N\}u_1$, where

$$LFT\{M,N\} = M_{11} + M_{12}N(I - M_{22}N)^{-1}M_{21}, \tag{0.10}$$

provided the inverse operator $(I - M_{22}N)^{-1}$ exists.
We call $M(s)$ the *coefficient matrix* of the linear fractional map.

Some Properties of Contractive and Unitary LFTs.
Given a linear fractional map as described above with $\det(I - M_{22}(\infty)N(\infty)) \neq 0$, the following hold:

1. If $\|M\|_\infty \leq 1$, then $\|N\|_\infty \leq 1$ implies that $\|LFT\{M,N\}\|_\infty \leq 1$.

2. If $M^T(-s)M(s) = I$ and $M_{21}(j\omega)$ is full row rank for all $\omega \in \mathbb{R}$, then $\|LFT\{M,N\}\|_\infty \leq 1$ if and only if $\|N\|_\infty \leq 1$.

The above results are due to Redheffer. For proofs and a discussion of these and related results, see [34].

State Space Realizations of FDLTI Systems.

In the Laplace Domain, a *state-space realization* of an $m \times p$ proper FDLTI system $G(s) \in \mathcal{R}$ is any set of four matrices $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times p}, C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times p}$ such that

$$G(s) = C(sI - A)^{-1}B + D.$$

We also use the notation

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

to denote such a realization.

- Given a matrix A associated with the realization of a transfer function matrix such as $G(s)$ as above, we say that A is stable if it has all eigenvalues in the open left half plane.
  If A is stable then $G(s)$ is stable in the sense that all its poles must be in the open left half plane.

- If $G(s)$ as described by the above state space realization is square and D is invertible, then the *inverse system* $G^{-1}(s)$ is well defined and has a state-space realization

$$G^{-1}(s) = \left(\begin{array}{c|c} A - BD^{-1}C & BD^{-1} \\ \hline -D^{-1}C & D^{-1} \end{array}\right). \tag{0.11}$$

- With $G(s)$ as described above, its *adjoint* system has state-space realization

$$G^T(-s) = \left(\begin{array}{c|c} -A^T & C^T \\ \hline -B^T & D^T \end{array}\right). \tag{0.12}$$

- Suppose one is given realizations of two transfer function matrices $G(s) = D_G + C_G(sI - A_G)^{-1}B_G$ and $H(s) = D_H + C_H(sI - A_H)^{-1}B_H$. If their product is well defined, it has a state-space realization

$$G(s)H(s) = \left(\begin{array}{cc|c} A_G & B_GC_H & B_GD_H \\ 0 & A_H & B_H \\ \hline C_G & D_GC_H & D_GD_H \end{array}\right). \tag{0.13}$$

  If their sum is well defined, it has a state-space realization

$$G(s) + H(s) = \left(\begin{array}{cc|c} A_G & 0 & B_G \\ 0 & A_H & B_H \\ \hline C_G & C_H & D_G + D_H \end{array}\right). \tag{0.14}$$

- Suppose one is given a pair of matrices (A, B), with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. The pair (A, B) is *controllable* if and only if the following matrix has full row rank for all $\lambda \in \mathbb{C}$

$$\left(\lambda I - A \quad B \right). \tag{0.15}$$

  This matrix has full row rank when $\Re\{\lambda\} \geq 0$ if and only if (A, B) is *stabilizable.*

- Suppose one is given a pair of matrices (C, A), with $C \in \mathbb{R}^{p\times n}$ and $A \in \mathbb{R}^{n\times n}$. The pair (C, A) is *observable* if and only if the following matrix has full column rank for all $\lambda \in \mathbb{C}$

$$\begin{pmatrix} \lambda I - A \\ C \end{pmatrix}. \tag{0.16}$$

  This matrix has full row rank when $\Re\{\lambda\} \geq 0$ if and only if (C, A) is *detectable.*

Invariant Zeros of Realizations of FDLTI Systems.

The *invariant zeros* of the realization of a transfer function matrix $M(s) = C(sI -$

$A)^{-1}B + D$ are the values of $\lambda \in \mathbb{C}$ at which

$$\text{rank}\left\{\begin{pmatrix} A-\lambda I & B \\ C & D \end{pmatrix}\right\} < \text{normrank}\left\{\begin{pmatrix} A-sI & B \\ C & D \end{pmatrix}\right\}. \tag{0.17}$$

In contexts where a particular realization of a transfer function matrix $M(s)$ is already implied, we shall often refer to the invariant zeros of that realization simply as the invariant zeros of $M(s)$.

Should the above realization of $M(s)$ have an invariant zero at $\lambda \in \mathbb{C}$, then it may arise due to any combination of the following:

- Should (A, B) have an uncontrollable mode $\lambda \in \mathbb{C}$, then these modes give rise to *input decoupling* zeros of $M(s)$.
- Should (A, C) have an unobservable mode $\lambda \in \mathbb{C}$, then these modes give rise to *output decoupling* zeros of $M(s)$.
- If the rank of $M(\lambda)$ is less than the normal rank of $M(s)$, then it is a *transmission zero* of $M(s)$.

Supposing D has at least as many rows as it has columns, then with λ a transmission zero of $M(s)$, there exist vectors $x_0 \neq 0$ and $\eta \neq 0$ such that

$$\begin{pmatrix} A-\lambda I & B \\ C & D \end{pmatrix}\begin{pmatrix} x_0 \\ \eta \end{pmatrix} = 0. \tag{0.18}$$

Algebraic Riccati equations.

Given matrices $A, R, Q \in \mathbb{R}^{n\times n}$ with $Q = Q^T, R = R^T$, consider the *algebraic Riccati equation (ARE)*:

$$A^T X + XA + XRX + Q = 0. \tag{0.19}$$

In general we shall be concerned with real symmetric solutions $X \in \mathbb{R}^{n\times n}$ of the ARE. We call such a solution X stabilizing if $\Re\{(A+RX)\} < 0$ and strong if $\Re\{(A+RX)\} \leq 0$. Associated with the Riccati equation (0.19) is the Hamiltonian matrix

$$H = \begin{pmatrix} A & R \\ -Q & -A^T \end{pmatrix}. \tag{0.20}$$

A matrix $M \in \mathbb{R}^{2n\times 2n}$ is said to be *Hamiltonian* if it satisfies the equality $JM = (JM)^*$ where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Since $J^* = J^{-1} = -J$, it follows that $JMJ^{-1} = -M^*$ and hence that for any eigenvalue $\lambda_i(M)$, $-\bar{\lambda}_i(M)$ is also an eigenvalue of M.

Lemma of Lyapunov

Lemma 0.0.1 *Suppose one is given the following linear equation*

$$A_1^T X + XA_2 + Q = 0_{m\times n}, \tag{0.21}$$

where $A_1 \in \mathbb{R}^{m \times m}$, $A_2 \in \mathbb{R}^{n \times n}$ *and* $Q \in \mathbb{R}^{m \times n}$.

1. *There exists a unique solution* $X \in \mathbb{R}^{m \times n}$ *if and only if* $\lambda_i(A_1) + \lambda_j(A_2) \neq 0$ *for all* i *and* j.

2. *In particular, suppose that* $Q = 0_{m \times n}$, *then if* A_1 *and* A_2 *have all eigenvalues in the open left half plane,* $X = 0_{m \times n}$ *is the unique solution of (0.21).*

3. *If* $Q > 0$ *and* $A_1 = A_2 = A$, *then* $X > 0$ *if and only if* $\Re\{\lambda(A)\} < 0$.

4. *If* $Q \geq 0$, $A_1 = A_2 = A$ *and* (Q, A) *is detectable, then* $X \geq 0$ *if and only if* $\Re\{\lambda(A)\} < 0$.

Part I

Control Design Context.

Chapter 1

Introduction.

On the distinction between Design and Synthesis.

In control system design, mathematical models of systems and signals are obtained and arranged in an attempt to reflect the essential features of the process to be controlled. Various theoretical and computational tools are then applied to these models, resulting in the synthesis of a feedback control law which, it is hoped, will meet engineering objectives.

In any given practical situation, the control design process involves a large input of resources and is not merely a computational task. Developing means for obtaining process models is in itself an area of intensive research which is not considered in depth here. It is a matter of judgement on the part of the designer as to how a given process is best modelled for the purpose of control design. The task of appropriately configuring the process and signal models in a manner which reflects the actual system and which respects the design objectives is not trivial. This activity requires considerable insight and experience and, invariably, the incorporation of process-specific information. The outcome is, hopefully, a well-defined *controller synthesis* problem in which the desired engineering criteria for selection of a controller have been expressed in mathematical terms. Different design methodologies, sets of specifications and types of process and signal models will give rise to their own classes of synthesis problems. A control system designer, having formulated a controller synthesis problem, must then draw upon the available mathematical synthesis tools to furnish a control law. The design engineer generally passes the appropriate data to a software implementation of a synthesis algorithm where a control law is computed, if one is available for the given data. A distinction is thus drawn in this thesis between the task of control system *design* and that of controller *synthesis*, which is just one aspect of the design process. This thesis is primarily concerned with controller synthesis. The results are presented in the trust that they will come to good use as additional tools in the process control designer's toolbox.

Key issues for Controller Synthesis.

Very often it will be necessary to iterate the design process until satisfactory closed loop performance is attained. In addition to further controller synthesis calculations, this iterative process may well include model refinement, adjustment of control objectives and possibly the reconfiguration of system models. One measure by which synthesis procedures can be judged is on the basis of their efficiency from the point of view of the design process. How easily can design objectives be cast in the mathematical framework of the synthesis algorithm? What is the computational burden associated with computing a control law? How easily can the synthesized control law be implemented? These questions are adopted as motivating principles in the present work, which seeks to develop results which are of relevance to controller synthesis with multiple objectives.

Chapter summary.

The purpose of this introductory chapter is twofold. Firstly some principles and results from control design and controller synthesis are reviewed. This establishes the context for the second part, in which the main topics of this thesis are introduced with the aid of a simple design example.

1.1 Control Design and Controller Synthesis.

In this section, the objectives of process control design are first reviewed. Some results of linear control design methodology are then summarized. A simple control design example is then presented; firstly with a view to making the background material concrete and secondly to help motivate the particular topics which are the focus of this thesis. Whilst being by no means exhaustive, it is hoped that the discussion in the present section will provide a reference point and an aid in the interpretation of the remainder of this thesis.

1.1.1 Elements of Control System Design.

Whilst most of the content of this thesis assumes that the reader has a fair degree of specialist knowledge, it is proper for the non-specialist reader to enquire as to the motivation and scope of the material presented. The intention in this subsection is to provide a non-technical summary of some key elements in control design, with a view to providing a framework for the technical details which follow. It is hoped that the present subsection will also be accessible to the non-specialist reader.

Figure 1.1 represents one way of thinking about an engineering system; from the point of view of *process control.* In *process control design* the focus is generally on the design of one subcomponent of the system, called the *controller.* The function of the controller is to *improve the quality of process behaviour* by making *on-line decisions* about a number

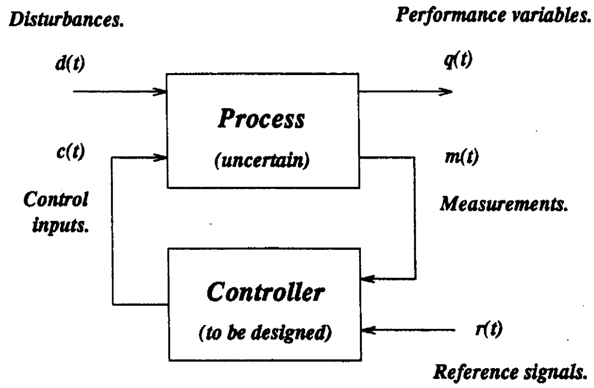

Figure 1.1: The process control task.

of process *control inputs* which can be freely chosen.

Remark: It should be noted that Figure 1.1 does *not* represent the generalized regulator configuration which has come to play a central role in recent linear controller synthesis results. It will be explained in due course how the generalized regulator configuration (which appears in Figure 1.5) relates to the above figure. □

Before embarking on the design of a controller, it is of benefit to identify how the various parts of the actual system may be classified according to the features identified in Figure 1.1.

The Process consists of all plant of interest, including actuators and sensors. Each of these components, including the instrumentation, may have dynamic behaviour. For the purpose of design, mathematical models of these components are generally adopted. These models are generally nonlinear and may also have time-varying elements. Physical and chemical laws, for example, invariably give rise to nonlinear features.

Importantly, in most engineering situations, there is some degree of *uncertainty* associated with these models. This may arise due to a combination of many factors; e.g. inadequacies in the scientific theory used to describe the system, use of simplified or reduced-complexity models, unforeseen variability in the process, the effect of off-line measurement errors in the modelling process, limited resources for the modelling process. Often, a *set* of process models $\mathcal{P}$ can be defined which can adequately capture the behaviour of the actual process.

Performance variables. A number of variables $q(t)$ associated with the process are deemed to quantify the performance of the process. These variables should relate directly to process operating specifications and or product quality goals. They need not be measureable on-line, however. They may instead be approximated from other process variables using models, which in turn form part of the overall process model.

Measurements. From the point of view of the controller, *measurements* $m(t)$ may be available of a number of variables related to the process. The process model generally contains a model of the measurement process, which describes the anticipated relationship between other process variables and the measured quantities.

Control inputs. A number of signals $c(t)$ associated with the process are generally considered to be *independently adjustable* by the controller. These signals exert some influence over process dynamics. The process model must describe how these signals affect other process variables, including the performance variables $q(t)$ and measurements $m(t)$.

Reference signals $r(t)$ are chosen by an external user as a means of expressing desired process performance objectives on-line. These signals are presented as an input to the controller. A decision has to be made by the designer as to the types of reference signals which the controller should be able to accommodate.

Disturbances. A process is generally subject to the influence of externally determined *disturbances* $d(t)$ which may affect the process behaviour in an undesirable way and which cannot be predicted in advance. In general, disturbances are not directly measureable. Disturbances may affect the process in any number of different ways; e.g. noise in sensors and actuators or direct perturbations on internal process variables. Disturbances are generally modelled as belonging to a *set* of possible signals which may be deterministic or probabilistic. The designer must assess in advance the likely nature of these disturbances. The process model must also include a description of the way in which disturbances are expected to influence the process dynamics.

The Controller generally consists either of analogue electronics hardware for which design specifications are sought, or a digital computer for which a program must be written. In either case, a *control law* is sought which accepts measurements $m(t)$ from the process and reference signals $r(t)$ from the user and computes the control signal $c(t)$ which is fed to the process.

Remark: Only *continuous-time* descriptions of signals and systems are considered. This assumption is not restrictive for many process-related systems and signals. However, it does pose some difficulties for controller implementation. Whilst direct analogue implementation of control laws is still important in practice, digital computers most often provide the hardware platform for control laws. Control algorithms therefore must be discrete-time in most instances. One technique for dealing with such situations is to design in continuous-time and then discretize the resulting controller. Simple discretization techniques which are satisfactory in many circumstances are outlined in standard textbooks (see e.g. [5]). □

Design Objectives.

Control laws are sought which ensure that the actual process performance variables $q(t)$ assume values which meet operating specifications. This means that they must respond in a specified manner to reference signals $r(t)$ in the face of disturbances $d(t)$, despite the fact that actual plant behaviour is uncertain.

Reference Tracking.
For a given class of reference signals $r(t)$, $q(t)$ should track $r(t)$ to within a specified tolerance.

Disturbance Attenuation.
The control law may also be required to attenuate the effect of disturbance signals on $q(t)$.

Stability.
High quality process performance should be sustainable, meaning that no process variable should be allowed to drift outside a safe operating range.

Feedforward and Feedback Control.

Assuming that a process model is available at the time of controller design, it is reasonable to expect that one could design a controller which has some *predictive* ability. If the model is sufficiently accurate, it may be possible to design a *feedforward* control law which has excellent reference following properties. (Feedforward controllers decide on control inputs only on the basis of the process model and reference inputs). Even with a sufficiently accurate process model however, feedforward control strategies provide no means of attenuating unmeasureable disturbances.

Control laws which also monitor measured process variables may be able to infer something about the *actual* disturbances, process behaviour and performance variables. If measurements are used in deciding inputs, then the the control law is said to use *feedback*. Feedback control laws are therefore able to attenuate disturbances; the influence of disturbances on measured signals may be detected and compensated for via adjustments to the control input variables. Feedback control laws result in a *closed loop* system system with its own *new* dynamical properties.

Robustness.

In the above discussion of design objectives, the issue of plant uncertainty was not addressed explicitly. Naturally, one would like to achieve the above closed loop design objectives for the actual process, despite the fact that there is some uncertainty in the process model.

Robustness of Control Laws.

A control law is termed *robust* if it is able to achieve closed loop performance objectives despite the variations in actual process behaviour from the model used to design the control law.

Some description of the *accuracy* of the model is invaluable in designing for good *actual* closed-loop behaviour. In robust control design, knowledge about the likely uncertainty in the process model is explicitly taken into account in the design process.

Robustness with respect to a Model Set.

A control law is said to be *robust with respect to the model set $\mathcal{P}$* if it is guaranteed to achieve closed loop performance specifications for all plants in $\mathcal{P}$.

One of the primary reasons for introducing feedback is that it is capable of reducing the sensitivity of closed loop performance to plant uncertainty. Feedback provides a means of reducing the effects of deviations of process behaviour from the model which was adopted at the time of control design. Feedback control laws systems have the potential to be *robust* since they can compensate for measured deviations from ideal plant behaviour.

In summary, mathematical models of processes and their uncertainty, together with on-line process measurements provide a powerful combination, the potential benefits of which are sought in process control design.

Trade-offs.

Some of the design and robustness objectives described above may well be in conflict. For example, improvement of reference tracking ability will often accentuate the undesirable effects of sensor noise on the closed-loop system.

1.1.2 Linear Control Design Methodology.

Figure 1.2 describes one means of addressing the control design task introduced in Figure 1.1. As a basis for the design of the controller, we take a *nominal* model $P(s)$ of the process which is a finite dimensional linear time invariant (FDLTI) system.[1] Also a FDLTI system $E(s)$ is introduced to assess the performance of the system (note that $E(s)$ is chosen to reflect the designer's objectives and does *not* model any part of the real process). For simplicity and ease of implementation, we also restrict our attention to the class of FDLTI control laws. This obviously rules out nonlinear and adaptive control strategies.

The Process Model Set $\mathcal{P}$.

It is well recognized that nominal linear models alone cannot always provide a descrip-

[1]For a summary of FDLTI systems, see the summary of **Notation, Definitions and Fundamental Results** at the beginning of this thesis.

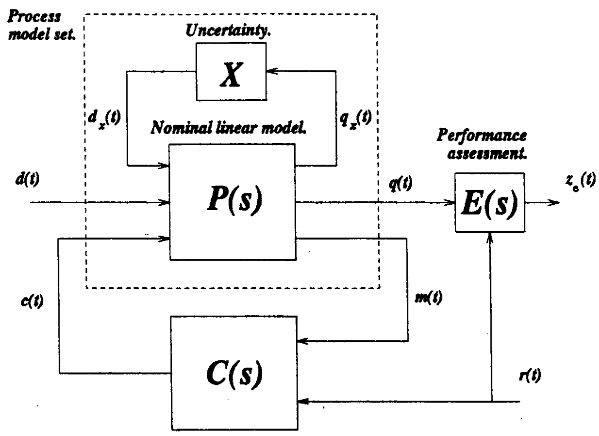

Figure 1.2: A framework for linear control design, including a nominal linear process model $P(s)$ with process uncertainty X (possibly nonlinear and time-varying), a linear control law $C(s)$ (to be designed) and a linear system $E(s)$ (to be chosen by the designer) which generates objective signals for performance assessment.

tion of the process which is adequate for control design. For example, it is well known that the actual closed-loop stability of *nominally* high-performing linear control designs can be very sensitive to deviations from the nominal plant model (see e.g. [34] and [26]).

In Figure 1.2, the process has been represented by a *set* of models $\mathcal{P}$ which consists of two components. The first is the nominal model P which is a FDLTI system. The second component is an *uncertain* dynamical system X (not necessarily FDLTI). With reference to Figure 1.2, observe that uncertain process behaviour is attributed to the system X which maps $q_x(t)$, consisting of signals related to the nominal process model, to $d_x(t)$ which consists of signals which disturb the nominal linear process dynamics. It is hoped that the process can be adequately described by $P(s)$, together with at least one $X \in \mathcal{X}$, where $\mathcal{X}$ is a set of systems which is chosen by the designer in an effort to capture his uncertainty in process behaviour. The elements P and $\mathcal{X}$ of the process model set $\mathcal{P}$ are described below in more detail.

Remark: A number of different ways of describing *how* the process uncertainty X perturbs the behaviour of the nominal model are commonly adopted. These include multiplicative, additive and normalized coprime factor uncertainty descriptions (see e.g. [34], [69],[82] and [66]). It is generally possible to describe uncertain plant behaviour in terms of a connection of the type shown in Figure 1.2 (see [69]). □

Nominal Linear Time Invariant Process Model.

The inputs of $P(s)$ include all process inputs (actuator signals, disturbances) as well as the the signal $d_x(t)$ which is the output of the model uncertainty. The outputs of $P(s)$ include all process outputs (sensor signals and performance variables), as well as the signal $q_x(t)$ which is the model uncertainty input.

Many analysis and synthesis tools for control design (including those developed in this thesis) draw upon the rich structure and many well-established results of linear systems theory. Working with linear systems does have some basis in practice as well as in mathematical expediency, however. Nominal linear models for process behaviour are generally obtained either by local linearization of nonlinear process models or from linear system identification experiments based on input/output data obtained during process operation. Also, it has become apparent in a number of nonlinear control design methodologies (see for example [39]) that a preliminary nonlinear transformation of process inputs and outputs can often be used to construct an equivalent linear description of a nonlinear process. Linear control techniques can often be used successfully in conjunction with this transformed system. The final result is a control law which is a combination of nonlinear and linear elements.

Process Uncertainty Set.

Linear models of time-varying or nonlinear systems are subject to the errors inherited from the nonlinear modelling process itself, from linearization of the nonlinear model and from experimental uncertainty. Thus the model uncertainty X will in general be a nonlinear time-varying operator. However, we consider here only one class of linear plant uncertainty description.

FDLTI Uncertainty.
It is assumed that $X \in \mathcal{X}$ is a FDLTI operator which is contained in a set $\mathcal{X}$, defined as follows:

$$\mathcal{X} = \{X(s) \mid X(s) = W_{dx}(s)X_n(s)W_{qx}(s) \text{ with } \|X_n\|_\infty < 1\}. \tag{1.1}$$

Thus the overall plant uncertainty X is described as a series connection of an unstructured normalized uncertainty $\|X_n\|_\infty \leq 1$, weighted by transfer function matrices on its inputs and outputs. Here $W_{dx}(s)$ and $W_{qx}(s)$ are stable, minimum phase *weighting functions* whose frequency response (magnitude, directions and phase) should be chosen to reflect process uncertainty bounds. They may, for example, be obtained using experimental frequency response data, perturbations in process model parameters (see the remark below), together with insights specific to the process at hand.

Remark: Another common plant uncertainty description is in terms of intervals on system parameters. System identification routines often calculate probability distributions for parameters as well as nominal estimates. Whilst a whole field of research in the linear systems area is devoted to the study of robustness to parametric uncertainties, it is also sometimes possible to translate such uncertainties into the frequency domain type described above. Given a nominal operating condition and nominal parameter values for a linearized process model, foreseeable variations in operating conditions or system parameters can often be used to generate linear frequency domain uncertainty bounds around the nominal model. For an application of this approach to robust Power System

Stabilizer design, see [17] and [101]. □

Objective Signals for Performance Assessment.

The FDLTI system $E(s)$ in essence compares the performance signals $q(t)$ and the reference signals $r(t)$ in some way, generating *objective signals* $z_e(t)$. Generally speaking, $E(s)$ is chosen in such a manner that making $z_e(t)$ *small* (in a sense to be defined) corresponds to the achievement of closed-loop dynamic performance objectives. The objective signals may, for example, consist of a deviation of a performance variable from a reference signal, the response of a performance variable to sensor noise or to other exogenous disturbances. For example, $E(s)$ may be a simple comparison $z_e(t) = r(t) - q(t)$ or perhaps a *model-matching* criterion $z_e(s) = q(s) - M_r(s)r(s)$. Here $M_r(s)$ is a transfer function matrix whose properties we would like the actual closed loop mapping from $r(t)$ to $q(t)$ to approximate. The object $E(s)$ may also include a *weighting function* $W_e(s)$ which emphasizes the frequency ranges where performance improvement is sought. For example $z_e(s) = s^{-1}(q(s) - M_r(s)r(s))$ will tend to emphasize the importance of steady-state tracking performance.

Linear Controller Structure.

The controller $C(s)$ is a FDLTI system with the following inputs and output:

$$c(s) = \begin{pmatrix} C_1(s) & C_2(s) \end{pmatrix} \begin{pmatrix} m(s) \\ r(s) \end{pmatrix}. \tag{1.2}$$

This is a general *two-degree-of-freedom* control law (see e.g. [64], [34] and [82]). In many design approaches it is stipulated that $C(s)$ have some additional internal structure. For example, a common assumption which corresponds to unity gain negative feedback is that $c(s) = K_1(s)(r(s) - m(s))$, where $K_1(s)$ is to be chosen. In this case the control law has the structure $C(s) = K_1(s)\begin{pmatrix} -I & I \end{pmatrix}$. Other examples include the observer/state-feedback structure [5] and the internal model control structure [82]. The controller $C(s)$ may also contain a loop shaping function (see e.g. [66], [64]). For simplicity, we do not consider control laws $C(s)$ with internal structure in this presentation.

A Design Example.

A design example is now introduced to illustrate the elements of Figure 1.2 which have been described above. The design example is depicted in Figure 1.3. It should be noted that all signals shown in this diagram are scalar and all systems are SISO. We shall return to the design example later in this chapter.

The process under consideration consists of two subsystems in cascade. The first subsystem consists of an actuator and some physical plant. The output $h_1(t)$ of subsystem 1 becomes the input of subsystem 2. The main quantity of interest is $h_2(t)$, the output of subsystem 2, which is to be regulated.

Noisy measurements $m_1(t)$ and $m_2(t)$ are available of $h_1(t)$ and $h_2(t)$, respectively.

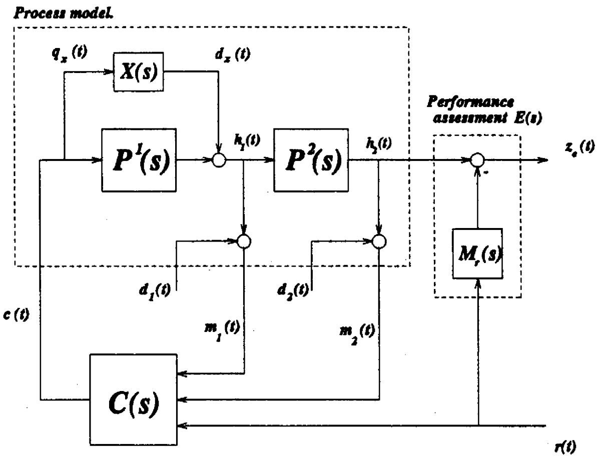

Figure 1.3: A two-degree-of-freedom linear control design example, including a nominal linear process model with additive uncertainty, measurement errors, and with an objective signal for model reference control.

Measurement noise is modelled via additive stationary stochastic disturbance signals $d_1(t)$ and $d_2(t)$. When compared with the bandwidth of the process and the anticipated bandwidth of the closed loop, the power spectral densities of both $d_1(t)$ and $d_2(t)$ can be considered as positive constants; say $\Phi_{d_1 d_1}(j\omega) = a_1^2$ and $\Phi_{d_2 d_2}(j\omega) = a_2^2$, for some constants $a_1 > 0$ and $a_2 > 0$.

The dynamical behaviour of the first subsystem is known to have a significant degree of uncertainty. It is modelled as a nominal FDLTI system $P^1(s)$, together with an additive FDLTI uncertainty $X(s)$. Subsystem 2 is modelled with a nominal FDLTI system $P^2(s)$. The uncertainty associated with subsystem 2 is known to be negligible when compared with that of subsystem 1. Suppose that the additive uncertainty $X(s)$ for subsystem 1 can be contained in a set $\mathcal{X}$, which is defined as follows:

$$\mathcal{X} = \{X(s) \mid X(s) = X_n(s)W_x(s) \text{ with } \|X_n\|_\infty < 1\}, \tag{1.3}$$

where $W_x(s)$ is a stable, minimum phase weighting function.

Suppose that $r(t)$ is expected to consists of a weighted sum of shifted unit step inputs, sufficiently separated in time. Let $M_r(s)$ be a FDLTI system whose unit step response is the desired step response of the actual closed loop mapping from $r(t)$ to $h_2(t)$ (see [64] where this approach is applied to control design for a distillation column example). One means of choosing $M_r(s)$ is as follows: first design a control law which uses only the nominal model and which achieves desirable nominal performance. Then choose $M_r(s)$ to be the nominal closed loop transfer function which results from this control law (see e.g. [101] where this approach is discussed).

A two-degree-of-freedom controller structure has been chosen which has access to the measured signals $m_1(t)$ and $m_2(t)$ as well as the reference signal $r(t)$, and generates the control input $c(t)$.

The main performance objective is then to ensure that for all $X \in \mathcal{X}$, the signal $h_2(t)$

responds to $r(t)$ in a manner which is *sufficiently close*[2] to the behaviour of $M_r(s)$. It should do this while maintaining closed loop stability and reasonable immunity to the sensor noises. □

In the design example discussed above, the notion of "closeness" of system signals was introduced. In order to make the process control design task more mathematically precise, we need to introduce some mathematical modelling of the disturbance and reference signals to which the system will be subjected.

Disturbance Signal Models.

The disturbance models adopted here take the form

$$d(s) = W_d(s)w_d(s), \tag{1.4}$$

where $W_d(s)$ is a FDLTI system and $w_d(s)$ is a *normalized* disturbance signal which is assumed to belong to a given signal space. The purpose of $W_d(s)$ is to capture the variation in frequency of the dominant magnitude and directions expected in the disturbance signals $d(t)$.

Weighted stochastic disturbances.
One approach is to describe disturbances as coloured white noise. In such a case, the disturbance signal is modelled as the response of a linear system $W_d(s)$ to a stationary unity variance white noise signal $w_d(t)$. Here $\Phi_{w_d w_d}(j\omega) = I$ and the system $W_d(s)$ is chosen to ensure that the power spectrum of the disturbance signal $d(t)$ reflects that expected in practice. Such signals are of prime interest in $\mathcal{H}_2$ controller synthesis, which will be later reviewed.

Weighted $\mathcal{L}_2$ disturbances.
Another common description of disturbance signals is in terms of weighted signals in $\mathcal{L}_2[0,\infty)$. Recall that $\mathcal{L}_2[0,\infty)$ consists of real valued signals defined on $t \in (-\infty,\infty)$ which are zero for $t \in (-\infty, 0)$ and for which $\|l\|_2 = \left\{\int_0^\infty l^T(t)l(t)dt\right\}^{\frac{1}{2}}$ is well defined. Such signals can be interpreted as having *finite energy* and may adequately capture certain types of process disturbances. One can also weight these signals; i.e. $d(s) = W_d(s)w_d(s)$ where $w_d(t) \in \mathcal{L}_2[0,\infty)$. $\mathcal{L}_2$ disturbances are also important in the development of $\mathcal{H}_\infty$ control design, as the $\mathcal{H}_\infty$ norm of a stable FDLTI system $M(s)$ is its induced norm with respect to $\mathcal{L}_2[0,\infty)$.

Deterministic Disturbances.
Another commonly adopted disturbance model is a deterministic linear system $W_d(s)$, perhaps with unspecified initial conditions or with a unit impulse input. The output of such a system can be used to model persistent deterministic signals such as an unknown

[2] Exactly what is meant by *close* is yet to be defined.

constant offset signal or a sinusoidal disturbance.

Sensor Noise Models.

Recall that in the design formulation adopted here, sensor noise signals are included in the disturbance $d(t)$. In designing a feedback law, the fact that only imperfect measurements are available must be taken into account. It is well known that noisy measurements impose a fundamental limitation on system performance (see for example [34]). A common means of modelling measurement errors is by additive weighted noise on the nominal plant output. In the design example introduced above, sensor noise was modelled using two broadband stationary stochastic signals.

Reference Signal Models.

The same approach can be taken to the modelling of reference signals as is taken to the modelling of disturbances. Reference signal models thus generally take the form

$$d(s) = W_r(s)w_r(s), \tag{1.5}$$

where $W_r(s)$ is a FDLTI system and $w_r(s)$ is a *normalized* disturbance signal of one of the types described above.

1.1.3 Generalized Linear Regulator Controller Synthesis Methodologies.

Having chosen nominal process models, uncertainty descriptions, controller structures, performance objective generating systems, disturbance and reference signal models, the *generalized linear regulator* provides a framework for unifying these elements into a single mathematical description of the design problem. The final step in the control design phase before controller synthesis is construction of the generalized plant. The generalized plant is an FDLTI system which consists only of known FDLTI systems. It therefore excludes the uncertain component of the plant uncertainty $X_n(s)$. The task of control design is then essentially reduced to the task of control law synthesis for the *generalized plant* $G(s)$ as depicted in Figure 1.4. This allows many different control design problems to be cast in the same mathematical framework which forms the basis for theoretical investigations and also makes possible general purpose software tools.

The Generalized Plant.

The FDLTI generalized plant G is given the following partitioning and interpretation of its input and output signals:

$$\begin{pmatrix} \tilde{z}(s) \\ y(s) \end{pmatrix} = \begin{pmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{pmatrix} \begin{pmatrix} \tilde{w}(s) \\ u(s) \end{pmatrix}, \tag{1.6}$$

where

- $\tilde{z}(t)$ is the generalized *objective output* signal.

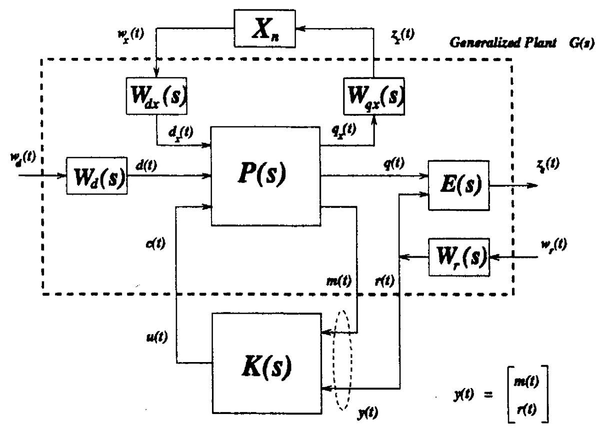

Figure 1.4: Inclusion of weighting functions for disturbances, model uncertainty and reference inputs, resulting in the *generalized plant.*

This may include a weighted combination of the following types of signals:
a) The normalized plant uncertainty input $z_x(t)$.
b) The closed-loop performance objective signals $z_e(t)$; e.g. tracking errors, control activity, response of certain variables to disturbances.

- $u(t)$ is the generalized *control input* signal.
This is the output of that part of the controller which is to be designed. It need not be the actuator input itself. However, the actuator input will be influenced by this signal (in a manner which depends on the chosen controller structure).

- $\tilde{w}(t)$ is the generalized *disturbance input* signal.
This may include a weighted combination of the following types of signals:
a) The normalized plant uncertainty output $w_x(t)$.
b) The normalized disturbance signal $w_d(t)$.
c) The normalized reference signal $w_r(t)$.

- $y(t)$ is the generalized *measurement output* signal.
This is the input of that part of the controller which is to be designed, it may consist of a combination of sensor outputs $m(t)$ and reference signal inputs $r(t)$. The chosen controller structure will influence the composition of $y(t)$.

The Closed Loop Operator with Uncertainty.

Given a control law $K(s)$, and some unstructured FDLTI normalized plant uncertainty $X_n(s)$ with $\|X_n\|_\infty < 1$, we define the *closed loop operator with uncertainty* as follows:

$$z_e(s) = S^{X_n,K}(s) \begin{pmatrix} w_d(s) \\ w_r(s) \end{pmatrix}, \tag{1.7}$$

where the operator $S^{X_n,K}(s)$ results from *simultaneously* connecting the normalized uncertainty $X_n(s)$ and the control law to the generalized plant in Figure 1.4. It is the designer's aim to choose $K(s)$ such that $S^{X_n,K}(s)$ meets design specifications for all possible X_n. A number of such synthesis objectives for $S^{X_n,K}(s)$ will be discussed in the next subsection.

The Nominal Closed Loop Operator and Controller Synthesis.

The effect of model uncertainty is generally not *explicitly* taken into account in controller synthesis for the generalized plant. A number of well known linear control design methodologies focus on synthesizing linear control laws for the nominal generalized plant only, without explicitly taking into account the uncertainty set in control law synthesis. This is the so-called *generalized regulator problem*, pictured in Figure 1.5. Whilst robustness properties are not explicitly treated using such an approach, if the generalized plant and synthesis objectives are chosen carefully, controllers with desirable robustness properties can be obtained. The main benefit of the $\mathcal{H}_\infty$ control design methodology is that it is able to give robustness guarantees for certain types of model uncertainty sets (not for the actual plant, however).

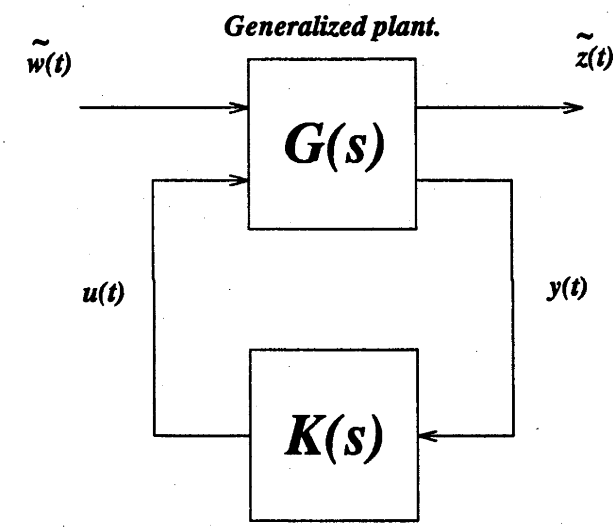

Figure 1.5: The generalized regulator problem.

A finite dimensional, linear, time-invariant, proper controller $K(s)$, when connected to $G(s)$ according to the control law $u(s) = K(s)y(s)$, results in the nominal closed loop operator

$$z(s) = T^K_{\tilde{z}\tilde{w}}(s)w(s), \tag{1.8}$$

where $T^K_{\tilde{z}\tilde{w}}(s)$ is given by

$$\begin{aligned} T^K_{\tilde{z}\tilde{w}}(s) &= LFT\{G(s),\ K(s)\} \\ &= G_{11}(s) + G_{12}(s)K(s)(I - G_{22}(s)K(s))^{-1}G_{21}(s). \end{aligned} \tag{1.9}$$

The notation LFT is short-hand for *linear fractional transformation.* The diagram 1.5 represents such a linear fractional transformation. The generalized plant is said to be the *coefficient matrix* in the linear fractional map.

The standard approach is to design a FDLTI controller $K(s)$ for the generalized plant, in the hope that obtaining certain closed loop properties of the *nominal* closed loop operator $T^K_{\tilde{z}\tilde{w}}(s)$ will ensure the desired objectives on the closed loop operator with uncertainty $S^{X_n,K}(s)$ and thereby the *actual* closed loop system. Often we shall deal with subsystems of the generalized plant which are associated with a selection of variables from $\tilde{w}$ and $\tilde{z}$. Let $w_j(t)$ and $z_i(t)$ be vectors consisting of a subsets of the vector signals $\tilde{w}$ and $\tilde{z}$, then $T^K_{z_i w_j}$ is referred to as *the closed loop operator with respect to the signal pair* $\{w_j, z_i\}$. The operator $S^K_{z_i w_j}$ can be defined in an analogous manner. Isolating particular subsystems of the closed loop operator $T^K_{\tilde{z}\tilde{w}}$ is particularly important from the perspective of multiple objective control.

State Space Realizations for Controller Synthesis.

It is often the case that state-space realizations of each constituent FDLTI system of the generalized plant are available. Combining these elements leads to a state-space realization of the generalized plant:

$$G(s) = \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right). \tag{1.10}$$

For computational purposes, this is generally the most favourable description of the generalized plant. We shall take such a realization as a basis for the development of all algorithms in this thesis.

A Generalized Plant for the Design Example.

We next show how a generalized plant may be constructed for the design example introduced in Figure 1.3.

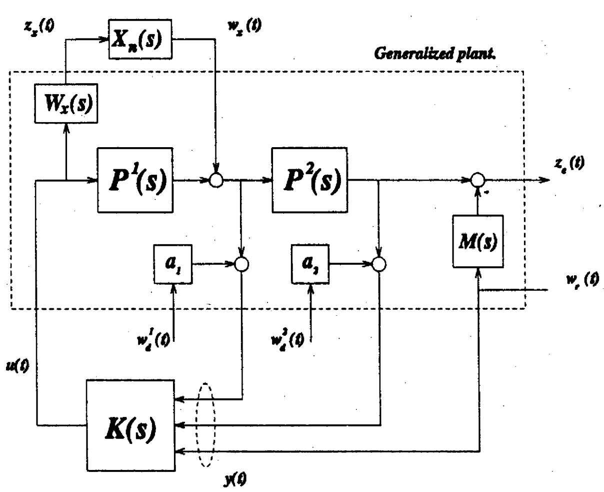

Figure 1.6: Construction of a generalized plant for the design example.

Recall first the definition of the model uncertainty set $\mathcal{X}$ in (1.3) which was defined in terms of a weighting function $W_x(s)$. We introduce the weighted signal $z_x(s) = W_x(s)q_x(s) = W_x(s)u(s)$ and incorporate the weighting function in the generalized plant. We now express some of the exogenous signals in Figure 1.3 in terms of *normalized* signals and weighting functions. Recall that the sensor noise signals $d_1(t)$ and $d_2(t)$ are white with $\Phi_{d_1 d_1}(j\omega) = a_1^2$ and $\Phi_{d_2 d_2}(j\omega) = a_2^2$, and where a_1 and a_2 are the standard deviations of the actual sensor noises. Observe in Figure 1.6 that these signals have been modelled as $d_1(t) = a_1 w_d^1(t)$ and $d_2(t) = a_1 w_d^2(t)$ where $w_d^1(t)$ and $w_d^2(t)$ are both unity variance white noise signals. Note that no other weights have been introduced in this example. For consistency, however, we introduce the following notation in constructing the generalized plant: $w_x(t) = d_x(t)$ and $w_r(t) = r(t)$.

Recall that the anticipated reference signals are step inputs. It might also be possible to describe this behaviour by incorporating a weighting function with an integrator at the reference input in the generalized plant (see e.g. [78] and [50]). However, the approach taken here (as in [64]) is to choose a reference model $M_r(s)$ which has a desirable unit step response. The aim is then to choose a control law such that $S^{X_n;K}_{z_e w_r}(s)$ is *close* to $M_r(s)$, in the hope that this will guarantee that the step responses will be close. In the next subsection we shall discuss in somewhat more detail some commonly adopted measures of closeness of FDLTI systems. The relationship between time domain and frequency domain response of a system is not easy to characterize. However it is possible to obtain some bounds on time responses given frequency response bounds; see e.g. [53] and [71].

Since a general two-degree of freedom control structure has been adopted, the control law $K(s)$ which is to be synthesized in Figure 1.6 is identical to $C(s)$ depicted in Figure 1.3. It also follows from this that the output of the control law to be synthesized is exactly $u(t) = c(t)$. Recall that this is not necessarily the case if the controller $C(s)$ contains internal structure. All signals available to $K(s)$ are combined as follows:

$$y(t) = \begin{pmatrix} m_1(t) \\ m_2(t) \\ w_r(t) \end{pmatrix}. \tag{1.11}$$

The controller to be designed thus has three-inputs and a single output.

Combining the plant model, reference model and weighting functions yields the following input/output description of the generalized plant:

$$\begin{pmatrix} z_e(s) \\ z_x(s) \\ \hline m_1(s) \\ m_2(s) \\ r(s) \end{pmatrix} = \left(\begin{array}{cccc:c} 0 & 0 & -M_r & P^2 & P^2P^1 \\ 0 & 0 & 0 & 0 & W_x \\ \hline a_1 & 0 & 0 & 1 & P^1 \\ 0 & a_2 & 0 & P^2 & P^2P^1 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right) \begin{pmatrix} w_d^1(s) \\ w_d^2(s) \\ w_r(s) \\ w_x(s) \\ \hline u(t) \end{pmatrix}. \tag{1.12}$$

□

1.1.4 Generalized Regulator Synthesis Objectives.

Having constructed the generalized plant, we now ask what *mathematical optimization criteria* should be introduced for the generalized regulator problem to express the desired performance specifications for the actual closed loop system.

Internal Stability.

The most basic objective is that the control law, when connected to the generalized plant results in a closed loop system which is internally stable. By this, we mean that the feedback connection of $G_{22}(s)$ and $K(s)$ is internally stable. (For a definition of internal stability, see the summary in the section **Notation, Definitions and Fundamental Results** at the beginning of this thesis.)

Remark:
It should be noted that internal stability of the interconnection of $G_{22}(s)$ and $K(s)$ does *not* always imply that the mappings from $\tilde{w}$ to internal loop variables are stable. For example, if the generalized plant contains weighting functions with imaginary axis poles (as in [50]) this will definitely not be the case. Henceforth, we shall assume that the generalized plant has only stable weighting functions. □

Define $\mathcal{K}$ to be the set of all internally stabilizing controllers for the generalized plant. Note that this is in general *not* the same as the set of all controllers which stabilize a given process, nor those which stabilize a given model set. Nevertheless, this set does play an important role since the minimum that any control law should do is stabilize the nominal plant. Nominal closed loop stability also plays an important role in establishing robust stability results using the small gain theorem (see e.g. [34]).

Operator Norms for Performance Assessment.

Here we consider controllers which either minimize or achieve a bound on a norm of selected submatrices of the *nominal* closed loop system:

$$\|T^K_{z_i w_i}\| \leq \gamma. \tag{1.13}$$

Here $z_i(t)$ and $w_i(t)$ consist of a *subset* of the elements of $\tilde{z}(t)$ and $\tilde{w}(t)$, respectively. We consider here only the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms for FDLTI systems. Formulations of the mathematical controller synthesis problems associated with these objectives are reviewed next.

1.1.5 Linear Quadratic Gaussian Control Design.

The purpose of the Linear Quadratic Gaussian (LQG) control design methodology is to minimize the effect on selected process objective signals of exogenous input signals which are described as stationary stochastic processes; e.g. process disturbances, measurement noise, actuator errors or reference signals.

The stationary stochastic disturbance signals are generally modelled as the response of stable weighting filters to a zero-mean unity variance white noise signal $n(t)$; $d(s) = W_d(s)n(s)$. Since the power spectral density matrix of $n(t)$ satisfies $\Phi_{nn}(j\omega) = I$, it follows that $\Phi_{dd}(\omega) = W_d(j\omega)W_d^T(-j\omega)$. Thus, if one has some prior knowledge of the power spectral matrix of the disturbance $\Phi_{dd}(\omega)$, the filter $W_d(s)$ should be chosen to be a *spectral factor* of $\Phi_{dd}(\omega)$ (for a definition of spectral factors see Chapter 6). The weighting function $W_d(s)$ is incorporated in the generalized plant. The normalized signal $n(t)$ is treated as a disturbance input to the generalized plant.

Any objective signal $e(t)$ can be weighted by a stable, minimum phase transfer function matrix, the singular values and directions of which can be chosen to reflect the relative importance of attenuation over different frequencies; $v(j\omega) = V(j\omega)e(j\omega)$. Assuming that $V(s)$ is invertible, then $\Phi_{ee}(\omega) = V^{-1}(j\omega)\Phi_{vv}(\omega)V^{-T}(-j\omega)$. The weighting function $V(s)$ also becomes part of the generalized plant.

The aim of LQG control is to minimize the total power of the weighted objective signal $v(t)$ in response to the stationary stochastic input $d(t)$. Equivalently, minimize the root mean square value of the signal $v(t)$ in response to the unity variance white noise input $n(t)$. Subject to such an input, the objective of LQG design is to minimize the following quantity:

$$\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T v^T(t)v(t)dt\right\}^{\frac{1}{2}} = \left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\text{trace}\{\Phi_{vv}(j\omega)\}d\omega\right\}^{\frac{1}{2}}, \tag{1.14}$$

which is the average root-mean-squared value of the weighted system objective signals $v(t)$.

Suppose that $T_{vn}^K(s)$ is the closed loop mapping between the normalized disturbance $n(t)$ and the weighted output $v(t)$. It is easy to show that $\Phi_{vv}(s) = T_{vn}^K(s)\ (T_{vn}^K)^T(-s)$ which again is due to the fact that $\Phi_{nn}(j\omega) = I$. This is equivalent to an $\mathcal{H}_2$ synthesis objective for the generalized plant, which we now review.

The $\mathcal{H}_2$ Controller Synthesis Problem.

Suppose one is given a generalized plant $G(s)$ with the following partitioning

$$\begin{pmatrix} v(s) \\ y(s) \end{pmatrix} = \begin{pmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{pmatrix}\begin{pmatrix} n(s) \\ u(s) \end{pmatrix} \tag{1.15}$$

with $v(t)$ the *controlled output* or *objective* signal of dimension n_v, $u(t)$ the *control input* signal of dimension n_u, $n(t)$ the *disturbance input* signal of dimension n_n and $y(t)$ the *measured output* signal of dimension n_y.

Linear, time-invariant, proper controllers $K(s)$ are sought which, when connected to $G(s)$ according to the control law $u = K(s)y$, result in an internally stable *closed-loop* *system*

$$T_{vn}^K(s) \quad = \quad LFT\{G(s),\ K(s)\} \tag{1.16}$$

such that the $\mathcal{H}_2$ norm is minimized:

$$\|T_{vn}^K\|_2 = \left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} \text{trace}\{T_{vn}^K(j\omega)(T_{vn}^K(-j\omega))^T\}d\omega\right\}^{\frac{1}{2}}. \tag{1.17}$$

Plant Uncertainty in LQG.

The LQG design methodology does not explicitly take into account plant uncertainty descriptions. Instead, the objective is to *optimally* solve the noise attenuation problem for the *nominal* linear generalized plant. Ideally, one would like to be able to make statements about the $\mathcal{H}_2$ norm of the uncertain operator $S_{vn}^{X_n,K}$ for all X_n such that $\|X_n\|_\infty < 1$. Results of this nature are not readily available by minimizing the $\mathcal{H}_2$ synthesis objective for the nominal generalized plant.

1.1.6 $\mathcal{H}_\infty$ Linear Robust Control Design.

The design of linear robust controllers is a broad topic (for a summary of different approaches see e.g. [65]). Generally a nominal linear plant model is first adopted, however various descriptions of plant uncertainty can arise and these lead to different design methodologies. Here we summarize a number of the robustness properties which can be obtained using the $\mathcal{H}_\infty$ design methodology and FDLTI uncertainty descriptions.

The symbol $\mathcal{H}_\infty$ represents the set of transfer function matrices which are analytic and uniformly bounded in the open right half plane. In $\mathcal{H}_\infty$ controller synthesis, control laws are sought which ensure that certain closed-loop transfer function matrices, associated with the generalized plant, lie in $\mathcal{H}_\infty$ and that the magnitude of their frequency response achieves a pre-specified bound. Let (w_i, z_i) be such a pair of signals associated with the generalized plant. The $\mathcal{H}_\infty$ synthesis objective is to ensure internal stability of the closed loop and that

$$\|T_{z_i w_i}^K\|_\infty < \gamma. \tag{1.18}$$

Careful choice of weighting functions in the generalized plant can be used to obtain frequency shaping in the actual closed loop system.

Remark: It should be recalled (see e.g. [34]) that with $G \in \mathcal{H}_\infty$, the norm $\|G\|_\infty$ is an induced norm with respect to $\mathcal{L}_2$ input and output signals. Thus if the disturbance signal $w_i(t)$ in the generalized regulator problem is an $\mathcal{L}_2$ signal,

$$\|z_i\|_2 < \|T_{z_i w_i}^K\|_\infty \|w_i\|_2. \tag{1.19}$$

Thus the $\mathcal{H}_\infty$ design objective has immediate interpretation in terms of the attenuation of finite energy signals. □

Robustness to Plant Uncertainty.

If weighted linear unstructured modelling errors are assumed, then robustness of the following closed-loop properties with respect to normalized FDLTI uncertainty X_n can be obtained using the $\mathcal{H}_\infty$ design methodology:

1. Internal Stability:
   $S^{X_n,K}$ is internally stable.

2. Reference tracking:
   $\|S^{X_n,K}_{z_e w_r}\|_\infty \leq 1$.

3. Disturbance attenuation.
   $\|S^{X_n,K}_{z_e w_d}\|_\infty \leq 1$.

Each of the above goals can be achieved by solving different manifestations of basically the same problem: *construct a controller which achieves specified closed loop frequency response magnitude bounds between appropriately selected input and output signals of the generalized plant.*

1. Internal Stability:
   A sufficient condition is to guarantee that both $T^K(s)$ is internally stable and $\|T^K_{z_x w_x}\|_\infty < 1$. This follows from the small gain theorem (see e.g. [34]).

2. Reference tracking:
   A sufficient condition is to guarantee that both $T^K(s)$ is internally stable and that

$$\left\|T^K_{z_{e,x} w_{r,x}}\right\|_\infty < 1, \tag{1.20}$$

   where

$$z_{e,x}(t) = \begin{pmatrix} z_e(t) \\ z_x(t) \end{pmatrix}, \tag{1.21}$$

$$w_{r,x}(t) = \begin{pmatrix} w_r(t) \\ w_x(t) \end{pmatrix}. \tag{1.22}$$

   This follows from Redheffer's results on contractive LFTs (See the section **Notation, Definitions and Fundamental Results** at the beginning of this thesis and the references [64] and [34]).

3. Disturbance attenuation.
   A sufficient condition is to guarantee that both $T^K(s)$ is internally stable and that

$$\left\|T^K_{z_{e,x} w_{d,x}}\right\|_\infty < 1. \tag{1.23}$$

   where

$$z_{e,x}(t) = \begin{pmatrix} z_e(t) \\ z_x(t) \end{pmatrix}, \tag{1.24}$$

$$w_{d,x}(t) \;=\; \begin{pmatrix} w_d(t) \\ w_x(t) \end{pmatrix}. \tag{1.25}$$

This also follows from Redheffer's results on contractive LFTs.

The $\mathcal{H}_\infty$ Controller Synthesis Problem.

Linear, time-invariant, proper controllers $K(s)$ are sought which, when connected to $G(s)$ according to the control law $u = K(s)y$, result in an internally stable *closed-loop system*

$$T_{zw}^K(s) \;=\; LFT\{G(s),\; K(s)\} \tag{1.26}$$

whose infinity norm is bounded by a given constant $\gamma > 0$:

$$\|T_{zw}^K\|_\infty < \gamma. \tag{1.27}$$

Remarks:

a) In other words, we seek controllers which are internally stabilizing and for which $T_{zw}^K(s) \in \mathcal{BH}_\infty^\gamma$. (Recall that we say $M(s) \in \mathcal{BH}_\infty^\gamma$ if $M(s) \in \mathcal{RH}_\infty$ and $\|M(s)\|_\infty < \gamma$.)
b) Henceforth, for brevity we call any $K(s)$ satisfying these conditions a *γ-admissible* or simply an *$\mathcal{H}_\infty$ controller.*
c) We denote the set of all linear $\mathcal{H}_\infty$ control laws as $\mathcal{K}^\gamma$. □

A number of different approaches to the $\mathcal{H}_\infty$ synthesis problem are available in the literature. The $\mathcal{H}_\infty$ synthesis problem is expressed in the frequency domain and the earliest development of the synthesis theory had such a focus. More recently, the problem has come to be formulated in the literature in terms of a state space description of the generalized plant. Controller existence conditions and formulae based on state-space computations are available. The state-space approach is summarized in Chapter 2. The fruit of research in this field over the past 5 to 10 years has been a collection of computational tools which can now be quite easily used to synthesize control laws which achieve this goal in many circumstances.

1.2 Motivation for and Summary of Thesis Topics.

This thesis contains a number of results relevant to state-space controller synthesis theory which are motivated by the earlier discussion on control design. The aim of this section is to explain these connections, with a view to making the material in subsequent chapters more accessible. The relevance of these results in the context of design is illustrated via the example which was introduced earlier in this chapter.

1.2.1 Assumptions on the Generalized Plant in Controller Synthesis.

Synthesis algorithms for $\mathcal{H}_\infty$ control laws have been derived in the literature under a variety of different assumptions on the generalized plant. In general, the more assumptions made, the more complete is the available synthesis theory. Whilst facilitating elegant solutions, assumptions tend to restrict the class of engineering design problems which can be addressed by the resulting synthesis theory. Thus the challenge for the control design engineer is to pose a controller synthesis problem (construct a generalized plant and choose optimization criteria) which is meaningful in the context of his design objectives, yet is amenable to the assumptions required by the currently available synthesis algorithms. The question as to the degree to which the assumptions made on the generalized plant are reasonable in an engineering design context is complex.[3] The generalized plant is a function of the design process in its many facets. The designer constructs a generalized plant according to his own design philosophy, the available process models, knowledge of external signals, chosen controller structure, chosen performance demands, anticipated plant uncertainty, to name only a few. Importantly in posing a synthesis problem, the designer is also restricted by the currently available synthesis algorithms. This is objectionable in that it may be necessary for him to somewhat artificially rearrange his mathematical description of the design objectives in order to use software synthesis tools. This is not in accord with the philosophy that the design process be driven primarily by the engineering objectives and not by the synthesis procedures. Mathematical insights can aid the designer, however they must be expressed in tangible and reasonable terms in the context of the design process.

In an effort to alleviate this restriction on the practitioner, one might seek a complete controller synthesis theory for an *arbitrary* FDLTI generalized plant. At least for a state-space formulation of the $\mathcal{H}_\infty$ problem, this seems on many accounts to be particularly difficult and is as yet unsolved. The task of the control engineering theoretician is to address synthesis problems which have been identified as being of practical importance. On this basis, it could be argued that only certain *subclasses* of generalized plants are likely to occur in practice and a synthesis theory appropriate to those problems should be sought. The following principle might be adopted: where design needs reveal deficiencies in current synthesis results, an attempt should be made to relax relevant assumptions. There have been a number of results of this nature in the literature in recent years (refer to Chapter 2 for a summary). This thesis presents easily implementable $\mathcal{H}_\infty$ controller synthesis results for a class of generalized plants which cannot be treated using commercially available synthesis software. The class of generalized plants considered here is characterized by the fact that more sensors and/or actuators are available than is normally assumed. Such *nonstandard* generalized plants can arise in different design contexts where there is some *redundancy* in the control law or in the measurements. In

[3]The following comment is made in Chapter 12 of [34]: "The central difficulty with using the generalized regulator to solve design problems is interfacing the engineering requirements to the mathematical optimization process."

the next section, we show how a nonstandard generalized plant arises when considering multiple objective robust control design for the design example which was introduced earlier in this chapter.

1.2.2 Towards Multiple Objective Robust Control.

We now address some of the issues which arise when one considers multiple synthesis objectives for a given generalized plant. This topic has been the subject of ongoing research for a number of years. The focus here is on synthesis with a mixed $\mathcal{H}_2$ / $\mathcal{H}_\infty$ objective.

Nonuniqueness of $\mathcal{H}_\infty$ Controllers.

It is well known that $\mathcal{H}_\infty$ controllers are in general *not unique*. On the one hand, one would like to find at least one solution to the $\mathcal{H}_\infty$ problem. On the other hand, it is also of interest to find a description of *all* $\mathcal{H}_\infty$ controllers which makes explicit any freedom in the choice of controllers. The possibility can then be investigated as to how this freedom might be used to further improve closed-loop system behaviour. That is, given the set of controllers which satisfy a given closed-loop $\mathcal{H}_\infty$ norm constraint, find a controller within this set which optimizes another, in general different, closed-loop performance measure.

A Mixed $\mathcal{H}_2$ / $\mathcal{H}_\infty$ Synthesis Problem.

We shall focus on the dual-objective generalized regulator problem described in figure 1.7. The inputs and outputs have been partitioned into two pairs (w, z) and (n, v). Two closed loop input/output objectives are sought in addition to the usual internal stability objective. The minimum possible $\mathcal{H}_2$ norm of T^K_{vn} is sought, subject to an $\mathcal{H}_\infty$ norm bound being satisfied on T^K_{zw}.

We now demonstrate the relevance of this problem in the context of the design example introduced earlier in this chapter.

Mixed $\mathcal{H}_2$ / $\mathcal{H}_\infty$ Objective for the Design Example.

It should be noted that the disturbance signals in the generalized plant of Figure 1.6 have been modelled using different classes of signals. The signals $w^1_d(t)$ and $w^2_d(t)$ are stationary stochastic processes with unity variance. The $\mathcal{H}_2$ optimization criterion is best suited to this type of disturbance signal. Recall that few statements can be made about robust $\mathcal{H}_2$ performance. The quantity we shall be concerned with here is simply $\|T^K_{z_e w_d}\|_2$, which is the *nominal* $\mathcal{H}_2$ performance, given a candidate control law $K(s)$.

The robust model matching and stability objectives are associated with the signals $w_x(t)$, $w_r(t)$, $z_e(t)$ and $z_x(t)$. The synthesis objective associated with these objectives is the satisfaction of an $\mathcal{H}_\infty$ norm bound on certain closed loop transfer function matrices.

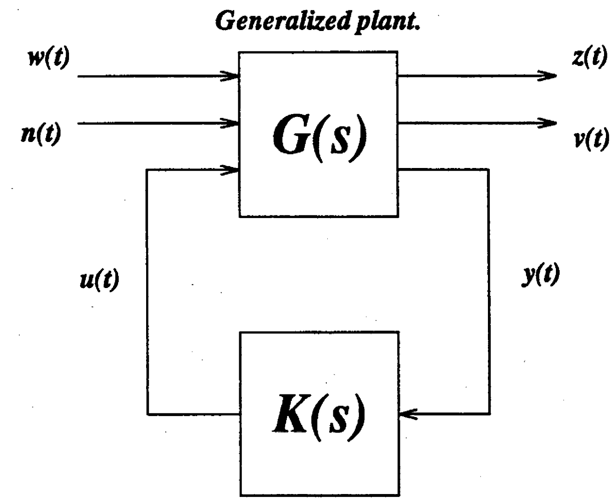

Figure 1.7: Multiple objective design framework with linear time invariant generalized plant and control law.

With the choice of variables

$$n(t) = \begin{pmatrix} w_d^1(t) \\ w_d^2(t) \end{pmatrix}, \tag{1.28}$$

$$v(t) = z_e(t), \tag{1.29}$$

$$w(t) = \begin{pmatrix} w_r(t) \\ w_x(t) \end{pmatrix}, \tag{1.30}$$

$$z(t) = \begin{pmatrix} z_e(t) \\ w_x(t) \end{pmatrix}, \tag{1.31}$$

the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ design objectives for the example being considered have been expressed in the framework introduced above.

This will guarantee the following robust stability and performance properties: for all X_n such that $\|X_n\|_\infty < 1$, $\|S_{z_e w_r}^{X_n,K}\|_\infty < 1$ and $S^{X_n,K}(s)$ will be internally stable.

One might ask what is the best nominal $\mathcal{H}_2$ performance that one could achieve with respect to the stationary stochastic inputs $w_d^1(t)$ and $w_d^2(t)$, whilst maintaining the $\mathcal{H}_\infty$ objective with respect to the other objectives. Note that we seek to *simultaneously* address the fundamentally different $\mathcal{H}_2$ and $\mathcal{H}_\infty$ synthesis criteria, corresponding to *nominal* sensor noise attenuation and the robust stability and reference tracking objectives.

The hope is that we can achieve the desired robust performance in the actual response of the variable $h_2(t)$ to step inputs $r(t)$, whilst ensuring the least possible influence of the two sensor noise signals. □

A Strategy for Solving the Mixed $\mathcal{H}_2$ / $\mathcal{H}_\infty$ Problem.

A strategy consisting of three steps is now proposed for solving the mixed $\mathcal{H}_2$ / $\mathcal{H}_\infty$ problem which was introduced in figure 1.7. This strategy does not constitute an immediately applicable synthesis algorithm since the third step in the procedure remains an unsolved problem. However, the first two steps of the procedure are addressed in this thesis.

Step 1: Construct the reduced-dimension generalized plant $\tilde{G}$ which has disturbance input $w(t)$ and objective output $z(t)$ only.

Step 2: Carry out an $\mathcal{H}_\infty$ synthesis for $\tilde{G}$, explicitly parametrizing the degrees of freedom in $\mathcal{H}_\infty$ control laws by deriving a state-space formula for all $\mathcal{H}_\infty$ controllers.

Step 3: Next, return to the original, larger generalized plant $G(s)$ and select the free parameters associated with $\mathcal{H}_\infty$ control laws for $\tilde{G}(s)$ in order to minimize the $\mathcal{H}_2$ norm of the closed loop transfer function which maps the stationary stochastic input $n(t)$ to the $\mathcal{H}_2$ performance objective $v(t)$.

Application of the Mixed $\mathcal{H}_2$ / $\mathcal{H}_\infty$ Strategy to the Design Example.

Application of the proposed mixed $\mathcal{H}_2$ / $\mathcal{H}_\infty$ synthesis strategy to the design example highlights a number of points which are of significance to the theoretical work found in the body of this thesis.

Step 1: The reduced-dimension generalized plant $\tilde{G}$ is as follows:

$$\begin{pmatrix} z_e(s) \\ z_x(s) \\ \hline m_1(s) \\ m_2(s) \\ r(s) \end{pmatrix} = \left(\begin{array}{cc:c} -M & P^2 & P^2P^1 \\ 0 & 0 & W_x \\ \hline 0 & 1 & P^1 \\ 0 & P^2 & P^2P^1 \\ 1 & 0 & 0 \end{array}\right) \begin{pmatrix} w_r(s) \\ w_x(s) \\ \hline u(t) \end{pmatrix} . \tag{1.32}$$

Step 2: The task of finding all $\mathcal{H}_\infty$ control laws for $\tilde{G}$ is not solvable via *standard* state-space methods. In particular, note that the number of measured signals $y(t)$ outnumbers the number of disturbances $w(t)$. A full parametrization of $\mathcal{H}_\infty$ control laws for such classes of generalized plants is derived in Chapters 3 and 4.

Step 3: The next task is to choose the free parameters in the description of all $\mathcal{H}_\infty$ control laws in a manner which minimizes the $\mathcal{H}_2$ norm of the closed loop transfer function mapping the stationary stochastic inputs $w_d^1(t)$ and $w_d^2(t)$ to the performance objective $z_e(t)$. This is a very hard problem. Just one aspect of the mixed $\mathcal{H}_2$ / $\mathcal{H}_\infty$ objective is treated in this thesis. An attempt is made to characterize a particular property of such control laws. In fact, it turns out that the closed loop transfer function matrix resulting from the optimal mixed control law *achieves* the $\mathcal{H}_\infty$ bound in most circumstances. Chapters 6 and 7 are concerned with developing an algorithm which allows this property to be characterized using state space realizations of a generalized plant together with a candidate control law.

□.

1.2.3 Summary of Thesis Topics and Contributions.

The general motivating principles for the topics addressed in this thesis can be summarized thus:

1. We need to relax mathematical assumptions in the formulation of synthesis problems in order to meet design needs.
2. A full description of control laws satisfying a given synthesis objective is needed if additional control objectives are to be considered.
3. A full understanding of the synthesis objective is needed before a synthesis theory can be developed.
4. All Algorithms should be as straightforward as possible to implement.

The research carried out in this thesis explores two main topics, which are outlined below. Each topic gives rise to specific computational and theoretical questions, which are pursued in subsequent chapters of this thesis.

1. **$\mathcal{H}_\infty$ Synthesis without Signal Dimension Restrictions (Chapters 2, 3 and 4).**
   A parametrization of all control laws is obtained for a class of state-space $\mathcal{H}_\infty$ synthesis problems. These results expand upon the so-called *standard* state-space $\mathcal{H}_\infty$ results obtained in [31], lifting restrictions on certain input and output signal dimensions which are introduced in the standard theory, but maintaining all other assumptions. This allows for the possibility that the number of control inputs exceeds the number of controlled outputs and/or the number of measured outputs exceeds the number of disturbance inputs. In such cases, the generalized plant is said to be *nonstandard.* As in the standard $\mathcal{H}_\infty$ problem, it is assumed that the direct feedthrough matrices from control inputs to controlled outputs and from measured outputs to disturbance inputs are full rank. Chapter 2 contains a summary of background material on state-space $\mathcal{H}_\infty$ synthesis and gives a precise formulation of the nonstandard $\mathcal{H}_\infty$ problems.

   With regard to the signal dimension assumptions made in the standard theory, the following observation was made in section 4.2.1 of [34]: "Since no measurement is error free and no control action is costless, these are reasonable assumptions." Whilst this statement is true for *single* objective problems, it is not necessarily true for *multiple* objective problems where different disturbances are associated with different synthesis criteria. This has been illustrated using the design example in the previous subsection. For that example it was shown that the $\mathcal{H}_\infty$ objective treated on its own is a *nonstandard* problem, there being less disturbances associated with the $\mathcal{H}_\infty$ objective than measurements. Similarly, a multiple objective synthesis problem may have objective signals which are associated with different synthesis criteria, in which case the number of objective signals associated with the $\mathcal{H}_\infty$ objective may be less than the number of control signals.

Nonstandard generalized plants may also arise in considering some single objective problems. For example it might be the case that some measurements are virtually error free. It might be argued in such a case that one could weight the noise inputs with a small parameter ϵ, resulting in a standard generalized plant. However, this will generally result in a poorly conditioned problem (see Chapter 3). In such a case, it would be of advantage to consider a generalized plant with noise-free measurements directly. Another example of a nonstandard $\mathcal{H}_\infty$ problem is the full information $\mathcal{H}_\infty$ problem where the number of measurements (state plus disturbances) clearly exceeds the number disturbances.

The nonstandard state-space $\mathcal{H}_\infty$ synthesis problem is addressed in Chapters 3 and 4 via two different methods. Both of the two methods draw upon standard $\mathcal{H}_\infty$ results, but use quite different techniques. Each approach results in $\mathcal{H}_\infty$ controller existence conditions and a full parametrization of $\mathcal{H}_\infty$ controllers. The results in each case are presented in terms of straightforward state-space calculations. The nonstandard $\mathcal{H}_\infty$ results obtained by each method are shown to be equivalent. The controller parametrizations contain additional degrees of freedom which are not present in the standard $\mathcal{H}_\infty$ results. This is important from the perspective of multiple objective robust control since the additional free parameters can be used to achieve closed-loop performance objectives in addition to the $\mathcal{H}_\infty$ constraint.

In the first of the two derivation methods, a family of augmented plants is introduced, to which the standard $\mathcal{H}_\infty$ theory can be directly applied. Subsequently, limiting arguments furnish existence conditions and controller parametrizations for the nonstandard generalized plant. In the second method, a lossless decomposition of the generalized plant is constructed, resulting in a so-called *temporary* generalized plant which is simpler and yet equivalent to the original generalized plant in that it has the same set of $\mathcal{H}_\infty$ control laws. The Youla parametrization of all stabilizing controllers is then applied to the temporary plant. This allows reduction of the problem to a simple so-called one-block $\mathcal{H}_\infty$ design problem which can be solved using standard $\mathcal{H}_\infty$ techniques.

The techniques used in the two methods of solving the nonstandard $\mathcal{H}_\infty$ problem have subsequently found successful application to other control problems. Similar techniques to those of the first method have been applied by other authors to an $\mathcal{H}_\infty$ servo problem (see [50]). The second approach has been applied to an $\mathcal{H}_2$ controller synthesis problem in which signal dimension restrictions are relaxed as for the nonstandard $\mathcal{H}_\infty$ problem. Chapter 4 concludes with controller parametrizations for $\mathcal{H}_2$ synthesis without signal dimension restrictions. The solution of this problem mirrors that for the $\mathcal{H}_\infty$ case in that it also has additional degrees of freedom. As in the standard case, the $\mathcal{H}_2$ algebraic Riccati equations are somewhat simpler than those for $\mathcal{H}_\infty$ synthesis. A proof of the nonstandard $\mathcal{H}_2$ results is reported elsewhere ([74]).

2. **The Boundary of the $\mathcal{H}_\infty$ Constraint and Algorithms for Spectral Factorization (Chapters 5,6 and 7).**

   The main purpose of this work is to investigate one aspect of the mixed $\mathcal{H}_2$ / $\mathcal{H}_\infty$ controller synthesis objective. Whilst a full synthesis procedure is not proposed, it is hoped that a closer understanding of the mixed design objective will aid in the development of synthesis algorithms. It turns out that one characteristic of control laws which minimize an $\mathcal{H}_2$ criterion subject to a closed-loop $\mathcal{H}_\infty$ constraint is that the closed loop transfer function matrix will generally be on the *boundary* of the $\mathcal{H}_\infty$ constraint set, in other words $\|T_{zw}^K\|_\infty = \gamma$. Chapter 5 contains a proof and discussion of this result. The $\mathcal{H}_\infty$ -norm bound on the closed-loop operator T_{zw}^K can be expressed in state-space terms via the *bounded real lemma*, which states that $\|T_{zw}^K\|_\infty \leq \gamma$ if and only if there exists a so-called strong solution of an associated algebraic Riccati equation (ARE). Study of the bounded real lemma has been instrumental in the development of the standard state space $\mathcal{H}_\infty$ controller synthesis theory (see e.g. [85]).

   Suppose one is given a candidate mixed $\mathcal{H}_2$ / $\mathcal{H}_\infty$ controller. In theory, one should be able to check the $\mathcal{H}_\infty$ constraint by solving for the strong solution of the algebraic Riccati equation associated with a state space realization of the closed loop system. The difficulty is that most algorithms for solving algebraic Riccati equations fail when the $\mathcal{H}_\infty$ norm of the closed loop system achieves the γ-bound. The main objective here is to find a simple and reliable algorithm for solving the ARE in such circumstances.

   The bounded real lemma is a special case of a broader class of state space spectral factorization problems with wider engineering significance. The difficulty in solving the algebraic Riccati equation when the γ-bound is achieved can be interpreted as being due to the fact that the realization of the associated spectral matrix has an imaginary axis invariant zero. An iterative algorithm for solving the ARE associated with this broader class of spectral factorization problems is developed in Chapters 6 and 7. The algorithm is guaranteed to converge at a known rate to the strong solution of the ARE even when the spectral matrix has imaginary axis invariant zeros.

   The proposed algorithm relies on bilinear transformation of a continuous time spectral matrix, to form a discrete time spectral matrix. The resulting realization of the discrete time spectral matrix has unit circle invariant zeros whenever the realization of the original continuous time spectral matrix has imaginary axis invariant zeros. The algebraic Riccati equation associated with the continuous time spectral matrix can be solved via a discrete algebraic Riccati equation associated with the discrete time spectral matrix. This is a well established technique for dealing with continuous time spectral factorization problems (see e.g. [4]). The algorithm for solving the continuous time problem relies on the development of an algorithm for solving the discrete time spectral factorization ARE. This algorithm is also of interest in its own right since it can be applied directly to spectral matrices arising in intrinsically discrete spectral factorization problems.

The discrete time spectral factorization algorithm relies on two novel developments. Firstly, it is shown that a Riccati difference equation associated with the discrete algebraic Riccati equation converges at a known rate of $\frac{1}{k}$ to the strong solution. Whilst many convergence results are available in the literature for RDEs, few results on convergence *rates* are available, particularly when the associated spectral matrix has unit circle invariant zeros. Secondly, a *doubling algorithm*, which calculates RDE iterates only at integral powers of two, is shown to be available for discrete time spectral factorization problems. Previously, doubling algorithms have only been demonstrated to be applicable to Kalman filtering and LQG problems. Combination of the RDE convergence rates and the doubling algorithm results in an algorithm for discrete spectral factorization which has known convergence properties.

Part II

State Space Controller Synthesis without Signal Dimension Restrictions.

Chapter 2

State-Space $\mathcal{H}_\infty$ Synthesis without Signal Dimension Restrictions.

Summary.

For an *arbitrary* state-space realization of a generalized plant $G(s)$, necessary and sufficient conditions for the existence of an $\mathcal{H}_\infty$ controller and a state-space description of all γ-admissible controllers are not available in the literature. However, full controller parametrizations have been derived in the literature under a number of different sets of assumptions on the state-space realization of the generalized plant. The state-space based results presented in [31] and [25] are perhaps the best known. We refer to the problem solved in these references as the *standard* $\mathcal{H}_\infty$ problem. One of the main purposes of the present work is to relax some assumptions which are made in the standard $\mathcal{H}_\infty$ synthesis theory concerning the relative dimensions of the signals in the generalized plant. With regard to $\mathcal{H}_\infty$ synthesis theory, the main emphasis of this thesis is therefore on a broadening of the class of generalized plants for which directly applicable state-space $\mathcal{H}_\infty$ synthesis algorithms are available.

The main aim of the present chapter is to formulate a state space controller synthesis problem without signal dimension restrictions which includes the standard and nonstandard cases. The controller synthesis problems for which the standard signal dimension restrictions are violated are referred to as *nonstandard.* Attention is given to the relevance of the nonstandard problems from the perspective of design, as described in Chapter 1. A summary of well known standard state space $\mathcal{H}_\infty$ results is given as well as a survey of the literature relevant to the nonstandard problems. It is shown in the two Chapters which follow this one that state-space existence conditions and full controller parametrizations similar to those for the standard problem are also available for nonstandard generalized plants. The nonstandard problems are addressed via two different routes in Chapters 3 and 4. A discussion and comparison of the results obtained using each approach appears in section 4.5 of Chapter 4.

2.1 Problem Formulation.

We now introduce the class of generalized plants for which we concentrate on developing $\mathcal{H}_\infty$ controller synthesis results. Henceforth, we assume that when realized in the Laplace domain, $G(s)$ has the following state-space structure:

$$G(s) = \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array} \right), \tag{2.1}$$

which satisfies assumptions **A.1**, **A.2** and **A.3** described below.

Assumptions on the Realization of $G(s)$.

A.1 (A, B_2) is stabilizable and (C_2, A) is detectable.

A.2 Both D_{12} and D_{21} are full rank.

A.3 Neither $G_{12}(s)$ nor $G_{21}(s)$, as described by the state space realization in (2.1), have imaginary axis invariant zeros.

It should be noted that assumption **A.2** above includes a broader class of assumptions than those which are treated in the standard case (which is described in detail in section 2.3). In particular, it is assumed *only* that the feedthrough matrices D_{12} and D_{21} are full rank and *no* assumptions are made regarding their *shape*; i.e. whether they have more columns than rows or vice versa. In the standard theory, the following two assumptions are made; the dimension of the objective signal $z(t)$ is at least as great as the dimension of the control input $u(t)$, and the dimension of the disturbance signal $w(t)$ is at least as great as the dimension of the measured signal $y(t)$. In subsection 1.2.2 of Chapter 1, a mixed $\mathcal{H}_2$ / $\mathcal{H}_\infty$ design example was introduced which gives rise to a generalized plant for $\mathcal{H}_\infty$ synthesis which violates the second of these signal dimension restrictions in the standard state-space $\mathcal{H}_\infty$ theory. We now state formally the $\mathcal{H}_\infty$ synthesis problem associated with the class of generalized plants described above, noting that this class of problems includes the standard problem as a special case.

The State-Space $\mathcal{H}_\infty$ Controller Synthesis Problem.
Given a realization of the generalized plant $G(s)$ in (2.1), satisfying **A.1**, **A.2** *and* **A.3**, *and a constant $\gamma > 0$, find necessary and sufficient conditions for existence, and a state space parametrization of all linear, time-invariant, causal controllers which result in an internally stable closed loop system $T_{zw}^K(s)$ for which*

$$\|T_{zw}^K\|_\infty < \gamma.$$

Remark: Recall from Chapter 1 that $T_{zw}^K(s)$ denotes the closed loop transfer function matrix which maps the generalized disturbance signal $w(s)$ to the generalized objective signal $z(s)$. □

2.2 On the Role of the Synthesis Assumptions in Design.

In summary, considerable creativity may be required of the control designer in constructing a generalized plant which satisfies the controller synthesis assumptions and yet adequately reflects the closed-loop design criteria. The present work aims at reducing this load on the designer by providing a controller synthesis theory which lifts the signal dimension restrictions in the standard theory.

The purpose of the present section is to investigate some of the implications of the assumptions **A.1**, **A.2** and **A.3** in the context of control design. Whilst these assumptions do define a broader class of generalized plants than that defined by the so-called standard assumptions, they still impose some limitations on the designer, which we now investigate. A number of these limitations have been addressed in the literature and the relevant references are given below.

Remarks on Assumptions A.1, A.2 and A.3.

a) If the generalized plant does not contain weighting functions which have closed right half plane modes, **A.1** says that there must exist a control law which stabilizes all unstable plant modes. It has be shown recently in [78] and [50] that **A.1** is violated in generalized plants arising in one formulation of $\mathcal{H}_\infty$ servo design problems. In such cases it may be that part of the disturbance signal $w(t)$ is the input of a weighting function which has imaginary axis modes. These modes will not be controllable from $u(t)$. Alternatively, it might be that part of the the objective signal $z(t)$ is the output of a weighting function which has imaginary axis modes. These modes will not be observable from $y(t)$.

b) Assumption **A.2** is quite a strong assumption in that it rules out some fairly common design problems. One example is designing for a bound on the sensitivity of a feedback system. Say one has $z(t) = y(t) = Pu(t) + w(t)$ where P represents the nominal linear plant dynamics and one wishes to design a control law $u(t) = Ky(t)$ which ensures both closed loop stability and $\|T_{zw}^K\|_\infty < \gamma$. Note here that T_{zw}^K is actually the closed loop sensitivity operator $(I - PK)^{-1}$. Note that if $P(s = \infty) = 0$ (which is often the case), one obtains $D_{12} = 0$. Whilst sensitivity minimization is rarely the only design objective, it is frequently one of a number of specifications on a closed loop system. This may cause D_{12} to lose rank.

c) In [99], existence conditions for a so-called *singular* $\mathcal{H}_\infty$ control problem are obtained in which assumptions **A.1** and **A.3** are maintained, whilst assumption **A.2** is removed. Whilst state-space controller existence results are found in [99] for such problems in terms of quadratic matrix inequalities, as yet there is no full description of control laws.

d) Now consider the implications of assumption **A.3**. Recall that the invariant zeros of

$G_{12}(s)$ can take the form of transmission zeros or of decoupling zeros [68]. It is possible to envisage design problems in which imaginary axis zeros of either of these types occur. Suppose, for example that assumption **A.1** is violated as described in **a)** above, being due to the existence of imaginary axis modes of A which are not controllable via B_2 (as in [78] and [50]). It is easy to check that this immediately gives rise to imaginary axis invariant (decoupling) zeros of the realization of $G_{12}(s)$. The case where $G_{12}(s)$ has imaginary axis *transmission* zeros has been addressed in [94]. □

Remarks on $G_{11}(\infty) = 0$ and $G_{22}(\infty) = 0$:
a) Note that we have assumed $G_{22}(\infty) = D_{22} = 0$ in (2.1). This assumption is generally not made at the expense of practical utility since in most situations it is generally the case that, due to physical considerations, the mapping from plant inputs to measurements is best modelled by a transfer function matrix $G_{22}(s)$ which is strictly proper.
b) The assumption that $G_{11}(\infty) = D_{11} = 0$ is somewhat stronger. It is certainly possible in generalized plants arising from genuine design problems that D_{11} is nonzero. In this case, the disturbance signal $w(t)$ influences the objective signal $z(t)$ directly (i.e. via D_{11}), in addition to possibly doing so indirectly via some dynamics (i.e. via transfer function $C_1(sI - A)^{-1}B_1$).
c) It has been shown in [92] that the assumptions $D_{11} = 0$ and $D_{22} = 0$ can be relaxed by a series of transformations, *provided* the standard dimensionality assumptions are made on D_{12} and D_{21}, in addition to the assumptions **A.1** ,**A.2** and **A.3** . Suppose one is given a generalized plant with D_{11} and/or D_{22} nonzero. A series of transformations is described in [92] which constructs a new generalized plant which satisfies $''D''_{11} = 0$ and $''D''_{22} = 0$ and which is equivalent to the original problem in the sense that a solution of the $\mathcal{H}_\infty$ problem for the new problem allows construction of a solution of the original problem. This process is also described in detail in [34].
d) The series of transformations referred to above in **c)** and described in [92] and [34] is not immediately applicable to all generalized plants in the class we consider here. It is a subject for further investigation as to whether a similar sequence of transformations is available which can be applied to generalized plants which satisfy **A.1** , **A.2** and **A.3** but violate the standard dimension restrictions. This issue is not investigated here; however a resolution of this would be important in making the synthesis results presented in this thesis applicable in design scenarios where $D_{11} \neq 0$ and/or $D_{22} \neq 0$. □

2.3 Standard State-Space $\mathcal{H}_\infty$ Controller Synthesis.

Recall that in the formulation of the $\mathcal{H}_\infty$ problem given above, no assumptions have been made concerning the relative number of inputs and outputs of the feedthrough matrices D_{12} or D_{21}. After a period of development over the past ten years or so, it has been found by a number of researchers that, provided in addition to **A.1** , **A.2** and **A.3** , the assumption **A.4** (stated below) is made, a check for existence and formulae for construction of $\mathcal{H}_\infty$ controllers is available which are based on straightforward state space calculations. Since this case has been extensively treated in the literature (see e.g. [31, 25, 32, 62, 92, 85]), we refer to it as the *standard* $\mathcal{H}_\infty$ problem. Henceforth we also refer to the assumptions, generalized plant and control laws in such cases as *standard.*

Standard Assumption on the Realization of $G(s)$.

A.4 D_{12} is *tall* $(n_z \geq n_u)$ and D_{21} is *fat* $(n_y \leq n_w)$.

Existence conditions and a formula for the construction of all $\mathcal{H}_\infty$ controllers for the standard case are summarized below. Historically, these results constitute a major step towards making the $\mathcal{H}_\infty$ design methodology computationally accessible. In this thesis, the standard state space results are taken as a basis for the development of a solution to the somewhat broader state space $\mathcal{H}_\infty$ synthesis problem which was formulated in section 2.1. Recall that this problem is broader than the standard problem in the sense that assumption **A.4** is *not* made.

Remark: Note that in the paper [25], slightly stronger assumptions are made (without loss of generality) on the realization of $G(s)$. In particular, it is assumed that $D_{12}^T\left(C_1 \;\; D_{12}\right) = \left(0 \;\; I\right)$ and that $D_{21}\left(B_1^T \;\; D_{21}^T\right) = \left(0 \;\; I\right)$. See also [34] where a detailed account is given as to why these assumptions introduce no loss of generality. □

A summary of state-space $\mathcal{H}_\infty$ results for the standard case is presented in Lemma 2.3.1. This result constitutes a *partial* solution to the $\mathcal{H}_\infty$ problem formulated in section 2.1. A state-space realization is given for a FDLTI system which becomes the coefficient matrix in a linear fractional transformation which parametrizes all controllers in terms of a free γ-bounded real transfer function matrix. The result in Lemma 2.3.1 is a minor variation on the standard $\mathcal{H}_\infty$ results presented in [31]. It is in one sense a weaker result than that of [31] since it is not assumed there that $D_{11} = 0$. However, in another sense it is a slightly stronger result than that of [31] in that a *family* of coefficient matrices is given for the linear fractional transformation, any one of which can be used to describe the set of all standard $\mathcal{H}_\infty$ controllers. This fact becomes important in the derivation of all nonstandard control laws presented in Chapter 3.

Lemma 2.3.1 *Given a generalized plant (2.1) satisfying assumptions* **A.1, A.2, A.3** *and* **A.4**, *a γ-admissible controller exists if and only if the following algebraic Riccati equations have stabilizing solutions $X \geq 0$ and $Y \geq 0$ which satisfy $\rho(XY) < \gamma^2$:*

$$
\begin{aligned}
0 &= X(A - B_2 D_{12}{}^\dagger C_1) + (A - B_2 D_{12}{}^\dagger C_1)^T X \\
&\quad + X(\gamma^{-2} B_1 B_1^T - B_2 E_{12}^{-1} B_2^T) X + C_1^T (I - D_{12} E_{12}^{-1} D_{12}^T) C_1,
\end{aligned} \tag{2.2}
$$

$$
\begin{aligned}
0 &= Y(A - B_1 D_{21}{}^\dagger C_2)^T + (A - B_1 D_{21}{}^\dagger C_2) Y \\
&\quad + Y(\gamma^{-2} C_1^T C_1 - C_2^T E_{21}^{-1} C_2) Y + B_1 (I - D_{21}^T E_{21}^{-1} D_{21}) B_1^T,
\end{aligned} \tag{2.3}
$$

with the definitions:

$$\begin{aligned} E_{12} &= D_{12}^T D_{12}, & E_{21} &= D_{21} D_{21}^T, \\ D_{12}{}^{\dagger} &= E_{12}^{-1} D_{12}^T, & D_{21}{}^{\dagger} &= D_{21}^T E_{21}^{-1}. \end{aligned}$$

Moreover, when the above conditions hold, all γ-admissible controllers for $G(s)$ are described as follows:

$$K(s) = LFT\{M(s), N(s)\} \tag{2.4}$$

where $N(s) \in \mathcal{BH}_{\infty}^{\gamma}$[1], and the coefficient matrix $M(s)$ has the following state-space realization:

$$M(s) = \left(\begin{array}{c|cc} \hat{A} + B_2 F_{\infty} + H_{\infty} \hat{C}_2 & -H_{\infty} & Z\hat{B}_2 V_{12} \\ \hline F_{\infty} & 0 & V_{12} \\ -V_{21}\hat{C}_2 & V_{21} & 0 \end{array} \right), \tag{2.5}$$

where V_{12} and V_{21} are any square matrices (not necessarily symmetric) which satisfy

$$V_{12} V_{12}^T = E_{12}^{-1}, \quad V_{21}^T V_{21} = E_{21}^{-1}, \tag{2.6}$$

and where

$$\begin{aligned} F_{\infty} &= -D_{12}{}^{\dagger} C_1 - E_{12}^{-1} B_2^T X, & \hat{A} &= A + \gamma^{-2} B_1 B_1^T X, \\ H_{\infty} &= Z(-B_1 D_{21}{}^{\dagger} - Y C_2^T E_{21}^{-1}), & \hat{C}_2 &= C_2 + \gamma^{-2} D_{21} B_1^T X, \\ Z &= (I - \gamma^{-2} Y X)^{-1}, & \hat{B}_2 &= \gamma^{-2} Y C_1^T D_{12} + B_2. \end{aligned}$$

Proof: The stated result has been established in [79] (see also [32]) for the case $\gamma = 1$ and with $V_{12} = E_{12}^{-\frac{1}{2}}$ and $V_{21} = E_{21}^{\frac{1}{2}}$. Here $E_{12}^{-\frac{1}{2}}$ and $E_{21}^{-\frac{1}{2}}$ are positive-definite symmetric square roots of E_{12}^{-1} and E_{12}^{-1}, respectively.

Generalization of the result in [79] to the case $\gamma \neq 1$ follows by applying the $\gamma = 1$ result to a generalized plant which has a realization which is the same as (2.1) except for the following replacements: $B_1 \to \gamma^{-1} B_1$ and $D_{21} \to \gamma^{-1} D_{21}$.

From the above two paragraphs, it follows that all $\mathcal{H}_{\infty}$ control laws are given by

$$K^{\#}(s) = LFT\{M^{\#}(s), N^{\#}(s)\} \tag{2.7}$$

where $N^{\#} \in \mathcal{BH}_{\infty}^{\gamma}$ is a free parameter and the coefficient matrix $M^{\#}(s)$ is given by (2.5) with the choice $V_{12} = E_{12}^{-\frac{1}{2}}$ and $V_{21} = E_{21}^{\frac{1}{2}}$.

Now let V_{12} and V_{21} be any square matrices which satisfy the identities (2.6) in the lemma statement. It follows that that $K^{\#}(s)$ can be rewritten as

$$K^{\#}(s) = LFT\{M(s), V_{12}^T E_{12}^{\frac{1}{2}} N^{\#}(s) E_{21}^{\frac{1}{2}} V_{21}^T\}, \tag{2.8}$$

where $M(s)$ is as given in (2.5) with the new values of V_{12} and V_{21}. Next observe that $V_{12} V_{12}^T = E_{12}^{-1}$ implies that $E_{12}^{\frac{1}{2}} V_{12} V_{12}^T E_{12}^{\frac{1}{2}} = I$. Since $V_{12}^T E_{12}^{\frac{1}{2}}$ is square, it is therefore an

[1]Recall that $\mathcal{BH}_{\infty}^{\gamma}$ is the set of all $M(s) \in \mathcal{RH}_{\infty}$ such that $\|M(s)\|_{\infty} < \gamma$

orthogonal matrix. It is easy to show that $V_{21}E_{21}^{\frac{1}{2}}$ is also orthogonal.

Next note that the transfer function matrix $N(s) = V_{12}^T E_{12}^{\frac{1}{2}} N^{\#}(s) E_{21}^{\frac{1}{2}} V_{21}^T \in \mathcal{BH}_\infty^\gamma$ if and only if $N^{\#} \in \mathcal{BH}_\infty^\gamma$. This follows trivially due to the orthogonality of the matrices which left and right multiply $N^{\#}$. The description of all controllers given in (2.4) follows. □

Remarks:

a) Note that stabilizing solutions of the $\mathcal{H}_\infty$ Riccati equations may exist which are not also nonnegative definite. This differentiates the $\mathcal{H}_\infty$ problem from the $\mathcal{H}_2$ problem where stabilizing solutions of the associated algebraic Riccati equations are generally nonnegative definite.

b) The standard *central* $\mathcal{H}_\infty$ controller is defined as the one which is obtained with the free parameter $N(s)$ set to zero. □

These results are important since they provide a computationally straightforward means for constructing control laws. Given generalized plant data, it is easy to test for the existence of $\mathcal{H}_\infty$ controllers and to calculate realizations of such controllers using standard linear algebra. This in turn makes translation into software straightforward. It is this attribute which we seek to maintain in developing a full solution of the state space $\mathcal{H}_\infty$ problem without signal dimension restrictions which was posed in section 2.1.

2.4 Signal Dimension Restrictions in the Standard Theory.

When compared with the standard problem, the significant feature of the $\mathcal{H}_\infty$ problem addressed in this thesis is that there are *no restrictions on the allowable dimensions of signals in the generalized plant.* Assumption **A.4** of the standard synthesis theory is violated for generalized plants in which control inputs outnumber objective signals and/or the number of disturbances associated with the infinity-norm objective is less than the number of measurable outputs. Such a generalized plant arose in the design considerations for the example in Chapter 1. The first implication for the control designer in such a circumstance is that the standard synthesis theory cannot directly furnish a control law for this generalized plant. An experienced designer may be able to re-cast the problem at hand into the standard framework, however this forces him to engage in work which is primarily due to theoretical limitations and not due to the overall design objectives.

It is evident that a nonstandard generalized plant should offer the control designer more flexibility in achieving closed-loop objectives than would be available for a comparable standard generalized plant with less measurements and/or control inputs. In order to utilize the expected redundancy in the control of nonstandard generalized plants to further improve system performance, a full parametrization of all internally stabilizing controllers which ensure the closed-loop $\mathcal{H}_\infty$ norm bound is required. The standard synthesis theory alone cannot provide such a parametrization. As was suggested in subsection 1.2.2 of Chapter 1, one might, for example, want to use the redundancy in the description of $\mathcal{H}_\infty$ control laws to minimize the nominal $\mathcal{H}_2$ performance of the closed loop system with respect to a second input/output signal pair.

Indeed, the parametrizations of control laws derived in Chapters 3 and 4 have *redundant* degrees of freedom which do not appear in the standard case. This redundancy is described in terms of free stable transfer function matrices which, whilst they do *not* influence the $\mathcal{H}_\infty$ control objective, do influence the closed loop dynamics and may therefore influence other closed loop transfer functions of interest.

In addition to their occurrence in applications, such nonstandard controller synthesis problems also have significance in the theoretical context. For example, the full information (FI) $\mathcal{H}_\infty$ control problem does not satisfy the standard assumption **A.4** since both the state and the disturbance signal are measurable and then $n_y = n + n_w > n_w$:

$$G^{FI}(s) = \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ I_n & 0 & 0 \\ 0 & I_{n_w} & 0 \end{array} \right). \tag{2.9}$$

A *particular* solution of the FI $\mathcal{H}_\infty$ problem plays an important role in the development of standard $\mathcal{H}_\infty$ results (see for example [25]). It is therefore somewhat curious that the standard theory cannot in turn be directly applied to describe all controllers for the full information problem. It has been recognized in [77] that the description of all FI $\mathcal{H}_\infty$ control laws presented in [25] was incomplete in that the redundancy in the parametrization was not included. A full parametrization is given in [77] in terms of an additional stable free transfer function matrix. A full solution to the FI problem is also given in [34] where this redundancy in the control law is also described.

2.5 Singular $\mathcal{H}_\infty$ Results without Signal Dimension Restrictions.

In [98] and [93], broader classes of problems are tackled than those in this thesis, in which no assumptions on the rank or relative number of inputs and outputs of the feedthrough matrices are made; assumption **A.2** is relaxed completely in addition to **A.4**, whilst assumptions **A.1** and **A.3** are maintained. The results of [98] and [93] are therefore directly applicable to the $\mathcal{H}_\infty$ problem treated in this section (where **A.1**, **A.2** and **A.3** hold).

Remark: The $\mathcal{H}_\infty$ problem treated in [98], is referred to as a *singular*. Whilst the assumptions adopted there include the particular problem addressed here, we refrain from calling the problem defined in section 2.1 singular due to lack of an appropriate interpretation in the present context. □

Controller Existence.

In [98], conditions for the existence of output feedback γ-admissible $\mathcal{H}_\infty$ controllers are established using a pair of quadratic matrix inequalities, the solutions of which allow state-space construction of one (but not all) controllers. In [93], existence conditions are obtained in terms of algebraic Riccati inequalities. Neither of these characterizations of existence are computationally easy to handle. Software for solving quadratic matrix inequalities is not in widespread use. It is demonstrated in the present work how questions of controller existence can be answered in terms of the solution of a pair of algebraic Riccati equations (AREs), rather than inequalities. In fact, in Chapter 4 the conditions given in [98] are applied directly to the class of plants defined in section 2.1, resulting in an expression of the existence conditions in terms of a pair of algebraic Riccati equations. It is precisely the additional assumption **A.2** (which was *not* adopted in [98]) which facilitates this.

Controller Parametrization.

In [99], under the same assumptions employed in [98], a parametrization of all *closed-loop transfer functions* is given which satisfy the $\mathcal{H}_\infty$ constraint. Parametrizations of all $\mathcal{H}_\infty$ *controllers* are not presented in either [98], [99] or [93] however. As is commented in [99], constructing such parametrizations appears to be a difficult problem for the general singular case and as yet no solution is available under the assumptions **A.1** and **A.3** alone.

Here, as in a number of recent papers (see [61], [76], [72], [73], [74], [75], [61], [46] and [107]) the standard assumption **A.4** is not made, whilst assumptions **A.1**, **A.2** and **A.3** are *maintained.* One of the main results of this thesis is that a full $\mathcal{H}_\infty$ controller parametrization *does* exist for the class of realizations of $G(s)$ given in (2.1) under the assumptions **A.1**, **A.2** and **A.3**. In a manner analogous to the standard case, explicit state-space parametrizations of all nonstandard $\mathcal{H}_\infty$ controllers are given in terms of the solutions of the AREs which are used to check controller existence conditions. These controller parametrizations contain free stable transfer function matrix parameters which are not present in the parametrization of $\mathcal{H}_\infty$ controllers for standard generalized plants. It should be emphasized once again that Assumption **A.2** plays an important role in developing these results.

2.6 The Nonstandard $\mathcal{H}_\infty$ Problems.

Recall that the solution of the standard $\mathcal{H}_\infty$ problem provides a partial solution to the state space $\mathcal{H}_\infty$ problem which was presented in section 2.1, since it solves that problem under the additional assumption **A.4**. In this section, we describe in more detail the three distinct cases of the state space $\mathcal{H}_\infty$ controller synthesis problem of section 2.1

which are *excluded* by this assumption. In the remainder of this chapter, a summary of the literature related to these problems is given, together with an outline of the way in which these problems have been addressed in this thesis.

The Standard Problem (Case 0).

We shall also refer to the standard problem as the Case 0 problem since it is a special case of the state space $\mathcal{H}_\infty$ problem defined in section 2.1.

Case 0 **A.4** holds; i.e. D_{12} tall, D_{21} fat: $(n_z \geq n_u,\ n_y \leq n_w)$

With respect to the matrices D_{12} and D_{21}, we say that they are *standard* in this case.

On the term *Nonstandard.*

It could be argued that *any* plant not simultaneously satisfying each of the assumptions **A.1**, **A.2**, **A.3** and **A.4** could validly be called nonstandard. However, in this thesis it is in a restricted sense that the term *nonstandard* is used; henceforth, we call a nonstandard $\mathcal{H}_\infty$ problem one for which assumptions **A.1**, **A.2** and **A.3** hold, but for which **A.4** is *violated.* We also refer to the assumptions and control laws in such cases as *nonstandard.* With respect to the matrices D_{12} and D_{21}, we refer them as being *nonstandard* in such circumstances.

The Nonstandard Problems.

In relaxing the standard assumption **A.4** on D_{12} and D_{21}, we allow for the possibility that either or both of these matrices are nonstandard; in other words $n_z < n_u$ and/or $n_w < n_y$. Note, however that we maintain **A.2**, the assumption that these matrices have full rank. It is a trivial observation that there are three distinct cases of the $\mathcal{H}_\infty$ problem defined in section 2.1 for which at least one of D_{12} and D_{21} violates **A.4**. These are listed below and have been denoted Cases 1, 2 and 3.

Case 1 Both D_{12} and D_{21} fat: $(n_z < n_u,\ n_y \leq n_w)$.

Case 2 D_{12} fat, D_{21} tall: $(n_z < n_u,\ n_y > n_w)$.

Case 3 Both D_{12} and D_{21} tall: $(n_z \geq n_u,\ n_y > n_w)$.

Cases 1 and 3 are called *singly-nonstandard* (i.e. only one of D_{12} and D_{21} is nonstandard) and case 2 is called *doubly-nonstandard* (i.e. both D_{12} and D_{21} are nonstandard).

2.7 Various Approaches to Nonstandard $\mathcal{H}_\infty$ Problems.

In this section, we summarize a number of different means of addressing nonstandard $\mathcal{H}_\infty$ problems.

Reduction to a Standard Problem by Ignoring some freedom in Controls/Sensors.

At this point, one might try dealing with nonstandard problems by effectively ignoring some freedom in the inputs or observations and applying standard $\mathcal{H}_\infty$ design techniques directly to the resulting modified generalized plant. For example, suppose one is given a problem with only D_{12} nonstandard, i.e. $n_z < n_u$, where the input signal $u(t) \in \mathbb{R}^{n_u}$. This can be transformed to a standard problem by defining a reduced-dimension input signal $u'(t) \in \mathbb{R}^{n_z}$, for example via the preliminary transformation $u(t) = D_{12}^T u'(t)$. The existence of a solution to the new problem is clearly sufficient for the existence of the original problem. Note, however that it will become apparent in Chapters 3 and 4 that the reverse is *not* true.

Conversion to a Standard Problem via Squaring Down Compensators.

A means for solving nonstandard problems is proposed in [61] where stable, minimum-phase squaring-down compensators allow transformation of the original non-standard $\mathcal{H}_\infty$ design problem into one where standard techniques are applicable. The paper [61] also contains a helpful discussion explaining why techniques employed in the solution of the standard problem are inapplicable in the nonstandard case.

The first step in the approach of [61] (see Lemma 4.1 of that paper) is to construct a state-space description of all stable closed loop systems $T_{zw}^K(s) = T_1(s) - T_2(s)Q(s)T_3(s)$ via the Youla parametrization, where $Q(s)$ is a free stable transfer function matrix and $T_1(s), T_2(s)$ and $T_3(s)$ are stable transfer function matrices for which state space realizations can be easily constructed, given a state space realization of the generalized plant. The result is an open-loop stable $\mathcal{H}_\infty$ synthesis problem where the Youla parameter $Q(s) \in \mathcal{RH}_\infty$ must be chosen as a control law to ensure $\|T_1(s) - T_2(s)Q(s)T_3(s)\|_\infty < \gamma$. It should be noted, however that the generalized plant associated with this new synthesis problem has state dimension twice that of $G(s)$.

At this point, the $\mathcal{H}_\infty$ synthesis problem is still nonstandard in that the matrix $T_1(s)$ $(T_3(s))$ will have nonstandard dimensions if the partition $G_{12}(s)$ $(G_{21}(s))$ of the original generalized plant has nonstandard dimensions. The abovementioned paper introduces a technique whereby the set of all closed loops can be reexpressed as $T_{zw}^K(s) = T_1(s) - \hat{T}_2(s)\hat{Q}(s)\hat{T}_3(s)$ where $\hat{T}_2(s)$ and $\hat{T}_3(s)$ are square transfer function matrices which are obtained from $T_2(s)$ and $T_3(s)$ via so-called *squaring down compensators*:

$$T_2(s)\hat{S}_{12}(s) = \begin{pmatrix} \hat{T}_2(s) & 0 \end{pmatrix}, \tag{2.10}$$

$$\hat{S}_{21}(s)\hat{T}_3(s) = \begin{pmatrix} \hat{T}_3(s) \\ 0 \end{pmatrix}, \tag{2.11}$$

where $\hat{T}_2(s)$ and $\hat{T}_3(s)$ are square transfer function matrices. The squaring-down compensators $\hat{S}_{12}$ and $\hat{S}_{21}$ for $T_2(s)$ and $T_3(s)$ are constructed using a state-space algorithm described in [61] which is based on Kalman's canonical form. (In this respect the authors of [61] demonstrate another rationale for choosing certain key matrices arising in the schemes described in Chapters 3 and 4.) The realizations of $\hat{T}_2(s)$ and $\hat{T}_3(s)$ have exactly the same set of unstable invariant zeros as $T_2(s)$ and $T_3(s)$. The result of this is that the task of selecting $\hat{Q}(s) \in \mathcal{RH}_\infty$ such that $\|T_1(s) - \hat{T}_2(s)\hat{Q}(s)\hat{T}_3(s)\|_\infty < \gamma$ is amenable to standard $\mathcal{H}_\infty$ results.

A characterization of all Youla parameters $Q(s) \in \mathcal{RH}_\infty$ which solve the original nonstandard $\mathcal{H}_\infty$ problem ($\|T_1(s) - T_2(s)Q(s)T_3(s)\|_\infty < \gamma$) is then obtained in terms of all solutions $\hat{Q}(s)$ to the intermediate standard $\mathcal{H}_\infty$ problem, free stable transfer function matrix parameters Q_1, Q_2 and Q_3, together with the dynamic squaring down compensators $\hat{S}_{12}$ and $\hat{S}_{21}$:

$$Q(s) = \hat{S}_{12} \begin{pmatrix} \hat{Q} & Q_1 \\ Q_2 & Q_3 \end{pmatrix} \hat{S}_{21}. \tag{2.12}$$

In [61] neither state space existence conditions nor explicit formulae for all controllers $K(s)$ of the original generalized plant are presented however. To obtain all control laws for the original nonstandard plant, one would need to insert the above subset of Youla parameters $Q(s)$ solving the intermediate problem into the description of all stabilizing controllers. If one were to write the resulting parametrization of control laws in linear fractional form, the resulting order of the coefficient matrix of the linear fractional map and therefore also of the central controller (i.e. that controller which results when all free parameters are set to zero) is likely to be much higher than the plant order, in which case model order reduction would be necessary.

In contrast, a formula is given in Chapters 3 and 4 for a central controller which has order *equal* to that of the generalized plant. Chapters 3 and 4 offer approaches which differ considerably from that adopted in [61]. The approach in Chapter 4 is similar to that of [61] in that it also draws upon the Youla parametrization. However, an important difference between the approach in [61] and that in Chapter 4 is that a so-called *lossless* decomposition of the nonstandard plant is employed. This decomposition facilitates the proof of the simple state-space formulae which are obtained.

Results reported for the Singly Nonstandard Cases.

In [46], controller parametrizations have been derived for the case 3 (singly) nonstandard problem under the additional condition that $G_{21}(s)$ has no zeros in the right half plane. This additional restriction is not imposed in this thesis. A parametrization of $\mathcal{H}_\infty$ controllers for case 3 plants is also given in [107], however without the assumption on the right half plane zeros of $G_{21}(s)$. The techniques of derivation in both [46] and [107] differ from the two alternatives presented in Chapters 3 and 4 of this thesis. Moreover, a full controller parametrization for the case of a *doubly nonstandard* plant (i.e. where both

D_{12} and D_{21} are nonstandard) is presented here, a result which is not trivially derivable from the singly nonstandard result.

2.8 Summary of the Nonstandard $\mathcal{H}_\infty$ Results in this Thesis.

Nonstandard $\mathcal{H}_\infty$ problems have been addressed in this thesis via two distinct routes. In both approaches, new $\mathcal{H}_\infty$ control problems are defined which are closely related to the nonstandard problem but satisfy the standard assumptions. Subsequently, standard controller existence and synthesis results are applied to the new standard problems, which can be shown to yield a solution to the original nonstandard problem. Both approaches have had outcomes of significance beyond the particular nonstandard $\mathcal{H}_\infty$ problem addressed here. In fact, the techniques of chapter 3 have been applied in [50] to an $\mathcal{H}_\infty$ servo problem, whilst the techniques of Chapter 4 have been applied in [74] to a nonstandard $\mathcal{H}_2$ problem, the results of which have been summarized in section 4.7.

Controller Existence Conditions.

In chapter 3, a family of (standard) augmented plants is constructed and limiting arguments are employed to establish existence conditions. In chapter 4, existence conditions are established using the singular $\mathcal{H}_\infty$ results of [98]. The same existence conditions are obtained by both approaches and are expressed in terms of a pair of algebraic Riccati equations and a coupling condition involving their solutions.

Controller Formulae.

A complete state space parametrization of all nonstandard $\mathcal{H}_\infty$ controllers, was first presented in [76]. The approach taken in [76] uses a combination of the Youla Parametrization and a lossless decomposition of the nonstandard plant. As in the standard case, this parametrization is expressed in terms of a linear fractional map with a fixed coefficient matrix and a parameter matrix consisting of a number of free transfer function matrices. One of these free transfer function matrices must be bounded real and the others are must be stable. The structure of the matrix of free transfer function parameters given in [76] was observed to be somewhat more complex and less symmetric than expected. In [75] and in Chapter 3 of this thesis, simpler and more symmetric controller parametrizations are obtained by applying limiting arguments to the state space controller formulae for the family of augmented plants which was used to find existence conditions. An awareness of the simpler form for controllers obtained in [75] prompted a reappraisal of the approach taken in [76], resulting in the work reported in [72] and in Chapter 4. The resulting complete nonstandard controller parametrization is expressed in terms of a matrix of transfer function parameters which has the same simple, symmetric structure as that obtained using the augmented plant approach.

The controller parametrizations obtained in Chapters 3 and 4 are completely equivalent in that they parametrize exactly the same set of control laws. However, the state-space construction of the coefficient matrices in the final linear fractional formula for all controllers differ slightly. It is shown in section 4.6.3 of Chapter 4 how this minor difference in the controller formulae can be accounted for.

Chapter 3

Nonstandard $\mathcal{H}_\infty$ Synthesis via Plant Augmentation.

Summary.

The nonstandard state space $\mathcal{H}_\infty$ problem is addressed in this chapter by augmenting the generalized plant to produce a family of standard generalized plants. Standard state-space $\mathcal{H}_\infty$ controller existence and parametrization results are then applied to the augmented plants. Subsequently, limiting arguments using well known results from analysis and linear algebra are applied to establish that the existence of nonnegative definite stabilizing solutions to two algebraic Riccati equations and satisfaction of an associated coupling condition are necessary and sufficient conditions for the existence of controllers. These results are analogous to the standard results, except for the fact that some preliminary state-space calculations must be performed before they can be applied. Limiting arguments are then applied to the parametrization of standard control laws for the augmented family of generalized plants to reveal a state-space parametrization of all controllers for the nonstandard problem. Controller formulae are presented for each of the three different cases of nonstandard generalized plants which were defined in section 2.6 of Chapter 2. However, only the doubly nonstandard case (case 2) is treated in detail since results for cases 1 and 2 can be obtained via reasoning entirely analogous to that presented for the doubly nonstandard case.

The technique employed here of defining a family of related standard problems promises to be applicable in relaxing other assumptions in the standard theory. In fact, similar techniques have been employed in [50] in obtaining $\mathcal{H}_\infty$ controller formulae for an $\mathcal{H}_\infty$ servo problem.

3.1 The Invariant Zeros of certain Subblocks of the Generalized Plant.

Recall the class of state space realizations of generalized plants which were introduced in Chapter 2:

$$G(s) = \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right). \tag{3.1}$$

The invariant zeros of the realizations of $G_{12}(s)$ and $G_{21}(s)$, as given by (3.1), are known to play a key role in feedback controller synthesis (see for example section 7.6 of [40]). The plant zeros are of interest since we are essentially interested in *regulation*, i.e. in making the signal $z(t)$ as close to zero as possible.

Recall that we make the following assumptions on the realization of $G(s)$:

A.1 (A, B_2) is stabilizable and (C_2, A) is detectable.

A.2 Both D_{12} and D_{21} are full rank.

A.3 Neither $G_{12}(s)$ nor $G_{21}(s)$, as described by the state space realization in (3.1), have imaginary axis invariant zeros.

Recall also the additional assumption which is made in the standard state-space $\mathcal{H}_\infty$ synthesis theory:

A.4 D_{12} is *tall* $(n_z \geq n_u)$ and D_{21} is *fat* $(n_y \leq n_w)$.

For convenience, we now list the four cases of the state-space $\mathcal{H}_\infty$ problem without signal dimension restrictions which was introduced in Chapter 2.

Case 0 **A.4** holds; i.e. D_{12} tall, D_{21} fat: $(n_z \geq n_u,\ n_y \leq n_w)$

Case 1 Both D_{12} and D_{21} fat: $(n_z < n_u,\ n_y \leq n_w)$.

Case 2 D_{12} fat, D_{21} tall: $(n_z < n_u,\ n_y > n_w)$.

Case 3 Both D_{12} and D_{21} tall: $(n_z \geq n_u,\ n_y > n_w)$.

The aim of this section is to investigate the invariant zeros of $G_{12}(s)$ and $G_{21}(s)$ under assumptions **A.1**, **A.2** and **A.3** and review means for their calculation. It will become apparent that assumption **A.2** has an important role in facilitating a simplified description of the invariant zeros of $G_{12}(s)$ and $G_{21}(s)$. We shall pay particular attention to the nonstandard cases 1–3 in which **A.4** is violated. These results will prove to be essential in each of the two developments of a nonstandard $\mathcal{H}_\infty$ synthesis theory which are presented in this thesis.

3.1.1 Invariant Zeros for Realizations of Nonsquare Transfer Function Matrices.

The following lemma characterizes the invariant zeros of any realization of a nonsquare transfer function matrix which is full rank at infinity.

Lemma 3.1.1

1. *Let $D \in \mathbb{R}^{l\times m}$ be of full column rank with $l > m$, and define $D^\dagger \in \mathbb{R}^{m\times l}$ and $D^\perp \in \mathbb{R}^{(l-m)\times l}$ according to the following equality:*

$$\begin{pmatrix} D^\dagger \\ D^\perp \end{pmatrix} (D \ \ (D^\perp)^T) = I_l. \tag{3.2}$$

Note also the following equality

$$DD^\dagger + (D^\perp)^T D^\perp = I_l. \tag{3.3}$$

Given the above definitions and a realization of a transfer function matrix

$$G(s) = C(sI - A)^{-1}B + D,$$

the invariant zeros of this realization are the unobservable modes of the pair $(A - BD^\dagger C,\ D^\perp C)$.

2. *Let $D \in \mathbb{R}^{l\times m}$ be of full row rank with $l < m$, and define $D^\dagger \in \mathbb{R}^{m\times l}$ and $D^\perp \in \mathbb{R}^{m\times(m-l)}$ according to the following equality:*

$$\begin{pmatrix} D \\ (D^\perp)^T \end{pmatrix} (D^\dagger \ \ D^\perp) = I_m. \tag{3.4}$$

Note also the following equality

$$D^\dagger D + D^\perp (D^\perp)^T = I_m. \tag{3.5}$$

Given the above definitions and a realization of a transfer function matrix

$$G(s) = C(sI - A)^{-1}B + D,$$

the invariant zeros of this realization are the uncontrollable modes of the pair $(A - BD^\dagger C, BD^\perp)$.

Proof: See [63]. An alternative proof of this result is contained in Appendix A. □

Remark: The matrices $D^\dagger$ and $D^\perp$ described in part **1** of the above lemma can be quite easily constructed using a QR decomposition of the matrix D (which is full column rank). Observe that such a decomposition has the form

$$D = QR = (\ Q_1 \quad Q_2\ ) \begin{pmatrix} R_1 \\ 0_{(l-m)\times m} \end{pmatrix} \tag{3.6}$$

with $Q \in \mathbb{R}^{l\times l}$ orthogonal and $R_1 \in \mathbb{R}^{m\times m}$ nonsingular. The choices $D^\dagger = R_1^{-1}Q_1^T$ and $D^\perp = Q_2^T$ then suffice. In part **2**, the same argument applied to D^T yields matrices $(D^\dagger)^T$ and $(D^\perp)^T$, where $D^\dagger$ and $D^\perp$ satisfy (3.4). □

3.1.2 Invariant Zeros for the Nonstandard Plants.

The lemma introduced in the previous subsection can be applied to the realizations of $G_{12}(s)$ and $G_{21}(s)$ associated with any nonstandard generalized plant (i.e. cases 1–3) as given in (3.1) which satisfies **A.2** . For convenience, we now summarize the consequences of Lemma 3.1.1 for nonstandard plants.

Lemma 3.1.2 *Given a realization of $G(s)$ as in equation (3.1) which satisfies assumptions* **A.1** , **A.2** *and* **A.3** *,*

1. *If D_{12} violates assumption* **A.4**, *then the invariant zeros of $G_{12}(s)$, as described by the state variable realization in (3.1), are given by the uncontrollable modes of the pair $(A - B_2 D_{12}{}^{\dagger} C_1, B_2 D_{12}{}^{\perp})$, where $D_{12}{}^{\dagger}$ and $D_{12}{}^{\perp}$ are given by application of (3.4) to D_{12}.*

2. *If D_{21} violates assumption* **A.4**, *then the invariant zeros of $G_{21}(s)$, as described by the state variable realization in (3.1), are given by the unobservable modes of the pair $(D_{21}{}^{\perp} C_2, A - B_1 D_{21}{}^{\dagger} C_2)$, where $D_{21}{}^{\dagger}$ and $D_{21}{}^{\perp}$ are given by application of (3.2) to D_{21}.*

Invariant zeros of the realizations of $G_{12}(s)$ and $G_{21}(s)$ as described above can arise in design problems in a number of different ways. Recall that assumption **A.3** precludes invariant zeros on the *imaginary axis.* Note that *right half plane* invariant zeros *are* permitted under the standing assumptions. Such invariant zeros may, for example, arise due to non-minimum phase process models within the generalized plant.

Note that even if one has a minimal realization (3.1) of the generalized plant $G(s)$, it may often be the case that the resulting realizations of $G_{12}(s)$ and $G_{21}(s)$ will be nonminimal. The realizations of $G_{12}(s)$ and $G_{21}(s)$ will then have invariant (decoupling) zeros. For example, recall that assumption **A.1** allows for the possibility of stable uncontrollable modes of (A, B_2) (due for example to a stable weighting function). It is easy to check that when (A, B_2) has uncontrollable modes, they are also uncontrollable modes of the pair $(A - B_2 D_{12}{}^{\dagger} C_1, B_2 D_{12}{}^{\perp})$ and, by Lemma 3.1.2, they are therefore invariant zeros of $G_{12}(s)$.

3.1.3 Canonical Forms for the Invariant Zeros of $G_{12}(s)$ and $G_{21}(s)$.

Two canonical forms and their corresponding state-space similarity transformations are now introduced which display information regarding the invariant zeros of $G_{12}(s)$ and $G_{21}(s)$. These canonical forms provide insight in the derivation of the existence conditions and controller parametrizations for nonstandard plants in both this chapter and in Chapter 4. Closely related observations have been made in [61].

Suppose D_{12} is nonstandard; then it follows from item *1* of Lemma 3.1.2 that the invariant zeros of $G_{12}(s)$ are the uncontrollable modes of the pair $(A - B_2 D_{12}{}^{\dagger} C_1, B_2 D_{12}{}^{\perp})$.

It is a standard result of linear systems theory that a state space similarity transformation T can be found which produces a controllability canonical form for the pair $(A - B_2 D_{12}{}^{\dagger} C_1, B_2 D_{12}{}^{\perp})$. For future reference, we now review how such a similarity transformation can be constructed. Let V_F denote a matrix of full column rank whose column space is the controllable subspace of the pair $(A - B_2 D_{12}{}^{\dagger} C_1, B_2 D_{12}{}^{\perp})$. Let U_F be any full rank matrix whose column space is complementary to that of V_F. It follows that

$$T = \begin{pmatrix} U_F & V_F \end{pmatrix} \tag{3.7}$$

defines a square, invertible state-space basis transformation. It is well known that using such a basis, one obtains a *controllability canonical form:*

$$T^{-1}(A - B_2 D_{12}{}^{\dagger} C_1)T = \begin{pmatrix} A_0 & 0 \\ A_{01} & A_1 \end{pmatrix}, \tag{3.8}$$

$$T^{-1} B_2 D_{12}{}^{\perp} = \begin{pmatrix} 0 \\ \beta_F \end{pmatrix}, \tag{3.9}$$

where $(A_1, \ \beta_F)$ is a controllable pair and, from Lemma 3.1.2, it follows that the modes of A_0 correspond to the invariant zeros of $G_{12}(s)$.

Suppose D_{21} is nonstandard; then it follows from item *2* of Lemma 3.1.2 that the invariant zeros of $G_{21}(s)$ are the unobservable modes of the pair $(D_{21}{}^{\perp} C_2, A - B_1 D_{21}{}^{\dagger} C_2)$. The dual of the controllability canonical form described above is the observability canonical form. We now review how a state space similarity transformation U can be found which produces this canonical form for the pair $(D_{21}{}^{\perp} C_2, A - B_1 D_{21}{}^{\dagger} C_2)$. Let V_H denote a matrix of full row rank whose row space is the controllable subspace of the pair $(D_{21}{}^{\perp} C_2, A - B_1 D_{21}{}^{\dagger} C_2)$. Let U_H be any full row rank matrix whose row space is complementary to that of V_H. Observe therefore that the following matrix is a square and invertible state-space transformation

$$U = \begin{pmatrix} U_H \\ V_H \end{pmatrix}. \tag{3.10}$$

It is well known that expressing the pair $(D_{21}{}^{\perp} C_2, A - B_1 D_{21}{}^{\dagger} C_2)$ in this basis via the following similarity transformation results in the *observability canonical form:*

$$U(A - B_1 D_{21}{}^{\dagger} C_2)U^{-1} = \begin{pmatrix} \alpha_0 & \alpha_{10} \\ 0 & \alpha_1 \end{pmatrix}, \tag{3.11}$$

$$D_{21}{}^{\perp} C_2 U^{-1} = \begin{pmatrix} 0 & \beta_H \end{pmatrix}. \tag{3.12}$$

The pair (β_H, α_1) is observable. From Lemma 3.1.2, we see that the invariant zeros of $G_{21}(s)$ are the eigenvalues of α_0.

3.2 A Parametrized Augmentation of the Nonstandard Plant.

Henceforth, we concentrate on the doubly nonstandard (case 2) generalized plant

$G(s)$. In this section we introduce a family of standard generalized plants $G^\epsilon(s)$ which is parametrized by a real number $\epsilon \geq 0$. The main conclusion of this section is that provided ϵ is small enough, the existence of a γ-admissible controller for $G^\epsilon(s)$ is equivalent to the existence of a γ-admissible controller for the nonstandard plant.

The ϵ-Augmented Plant.

Consider a family of generalized plants $G^\epsilon(s)$ with the following state-space description:

$$G^\epsilon(s) \triangleq \left(\begin{array}{c|ccc} A & B_1 & \epsilon\bar{B}_1 & B_2 \\ \hline C_1 & 0 & 0 & D_{12} \\ \epsilon\bar{C}_1 & 0 & 0 & \epsilon({D_{12}}^\perp)^T \\ C_2 & D_{21} & \epsilon({D_{21}}^\perp)^T & 0 \end{array}\right) = \left(\begin{array}{c|cc} A & B_1^\epsilon & B_2 \\ \hline C_1^\epsilon & 0 & D_{12}^\epsilon \\ C_2 & D_{21}^\epsilon & 0 \end{array}\right). \tag{3.13}$$

Recall that D_{12} is fat and D_{21} is tall in the doubly nonstandard case and the matrices ${D_{12}}^\dagger, {D_{12}}^\perp, {D_{21}}^\dagger$ and ${D_{21}}^\perp$ are defined by the following relations:

$$\begin{pmatrix} D_{12} \\ ({D_{12}}^\perp)^T \end{pmatrix} \begin{pmatrix} {D_{12}}^\dagger & {D_{12}}^\perp \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \tag{3.14}$$

$$\begin{pmatrix} {D_{21}}^\dagger \\ {D_{21}}^\perp \end{pmatrix} \begin{pmatrix} D_{21} & ({D_{21}}^\perp)^T \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{3.15}$$

Refer to section 3.1 for a description of how the above matrices may be constructed. Appropriate choices for the matrices $\bar{B}_1, \bar{C}_1$ in (3.13) will become clear in the ensuing analysis.

We have thus augmented the original generalized plant with extra disturbance and objective signals. We do this with the aim of creating augmented generalized plants which satisfy the standard assumption **A.4**, in fact this actually renders the feedthrough matrices D_{12}^ϵ and D_{21}^ϵ square and full rank. The input/output behaviour of $G^\epsilon(s)$ can be written as follows:

$$\begin{pmatrix} \tilde{z}^\epsilon(t) \\ \bar{z}^\epsilon(t) \\ y^\epsilon(t) \end{pmatrix} = G^\epsilon(s) \begin{pmatrix} \tilde{w}^\epsilon(t) \\ \bar{w}^\epsilon(t) \\ u^\epsilon(t) \end{pmatrix}, \tag{3.16}$$

$$\begin{pmatrix} z^\epsilon(t) \\ y^\epsilon(t) \end{pmatrix} = G^\epsilon(s) \begin{pmatrix} w^\epsilon(t) \\ u^\epsilon(t) \end{pmatrix}, \tag{3.17}$$

where the signals $y^\epsilon(t)$ and $u^\epsilon(t)$ have the same dimensions as $y(t)$ and $u(t)$, respectively and the signals $\tilde{z}^\epsilon(t)$ and $\tilde{w}^\epsilon(t)$ have the same dimensions as $z(t)$ and $w(t)$, respectively. The signals $\bar{z}^\epsilon(t)$ and $\bar{w}^\epsilon(t)$ have dimensions $n_u - n_z$ and $n_y - n_w$, respectively.

Satisfaction of the Standard Assumptions for the Augmented Plants.

The augmented feedthrough matrices D_{12}^ϵ and D_{21}^ϵ are square, of full rank and it follows from (3.14) and (3.15) that when $\epsilon > 0$, they have inverses:

$$\begin{pmatrix} D_{12} \\ \epsilon({D_{12}}^\perp)^T \end{pmatrix}^{-1} = \begin{pmatrix} {D_{12}}^\dagger & \frac{1}{\epsilon}{D_{12}}^\perp \end{pmatrix}, \tag{3.18}$$

$$\begin{pmatrix} D_{21} & \epsilon(D_{21}{}^{\perp})^T \end{pmatrix}^{-1} = \begin{pmatrix} D_{21}{}^{\dagger} \\ \frac{1}{\epsilon} D_{21}{}^{\perp} \end{pmatrix}. \tag{3.19}$$

This choice of D_{12}^{ϵ} and D_{21}^{ϵ} ensures that assumptions **A.2** and **A.4** hold for $G^{\epsilon}(s)$. Assumption **A.1** holds trivially. In a later section it is shown that $\bar{B}_1$ and $\bar{C}_1$ can be chosen in a manner which ensures that $G^{\epsilon}(s)$has no imaginary axis invariant zeros, thus guaranteeing **A.3**. It follows that, for any nonzero ϵ, the standard state-space $\mathcal{H}_{\infty}$ theory summarized in section 2.3 can be directly applied to the realization of $G^{\epsilon}(s)$described above. It will be shown that this fact, in conjunction with the observations in the remainder of this subsection, allows the deduction of both existence conditions and a full parametrization of nonstandard controllers for $G(s)$.

Remark: Augmentation of a generalized plant has also been used in [21] for treating certain *singular* $\mathcal{H}_{\infty}$ problems. When compared with the approach adopted here, there are substantial differences in the problem treated, the techniques used and the results obtained. For the problems treated in [21], D_{12} and D_{21} may violate **A.2** (i.e. be rank deficient) but they are assumed to be standard (i.e. they satisfy **A.4**). In [21], D_{12} is tall and has its rows further augmented by a matrix ϵI_{n_u}, whilst D_{21} is fat and has its columns further augmented by a matrix ϵI_{n_y}. In [21] the matrices B_1 and C_1 are augmented with $\bar{C}_1 = 0$ and $\bar{B}_1 = 0$. The fact that we take these quantities to be nonzero in our approach is a key to achieving the main results. □

Relating $\mathcal{H}_{\infty}$ Controllers for the Original and Augmented Plants.

We now make two key observations regarding the augmented plant which lead to Lemmas 3.2.1 and 3.2.2 and thus to Theorem 3.2.1.

1. *Internal stability.* Observe that $G_{22}^{\epsilon}(s) = G_{22}(s)$. If $K(s)$ internally stabilizes $G_{22}(s)$, then by [33] (Ch.4 Thm.1), it will internally stabilize both $G^{\epsilon}(s)$ and $G(s)$.

2. *Augmentation in closed-loop.* Given a linear control law $K(s)$, observe the following relationship between the augmented and original closed-loop transfer matrices

$$T_{z^{\epsilon}w^{\epsilon}}(s) = \begin{pmatrix} T_{\tilde{z}^{\epsilon}\tilde{w}^{\epsilon}}(s) & T_{\tilde{z}^{\epsilon}\bar{w}^{\epsilon}}(s) \\ T_{\bar{z}^{\epsilon}\tilde{w}^{\epsilon}}(s) & T_{\bar{z}^{\epsilon}\bar{w}^{\epsilon}}(s) \end{pmatrix} = \begin{pmatrix} T_{zw}(s) & \epsilon(\times) \\ \epsilon(\times) & \epsilon^2(\times) \end{pmatrix}, \tag{3.20}$$

where $\times$ denotes fixed ϵ-independent transfer function matrices which are not of immediate interest. Here $T_{zw}(s)$ is the closed loop transfer function mapping $w(t)$ to $z(t)$ which results from connecting the *same* controller $K(s)$ to the original (unaugmented) nonstandard generalized plant.

Lemma 3.2.1 *Given the plant $G(s)$ in (3.1), suppose that for some $\epsilon > 0$, there exists a γ-admissible controller $K^{\epsilon}(s)$for $G^{\epsilon}(s)$as given in (3.13), then $K^{\epsilon}(s)$is γ-admissible for $G(s)$.*

Proof: Recall observations *1* and *2* which were made immediately prior to the lemma statement. Since $K^\epsilon(s)$internally stabilizes $G^\epsilon(s)$, by observation *1* it also internally stabilizes $G(s)$. By hypothesis, $K^\epsilon(s)$also ensures $\|T_{z^\epsilon w^\epsilon}\|_\infty < \gamma$. From observation *2* note that $T_{zw}(s)$ is a submatrix of $T_{z^\epsilon w^\epsilon}(s)$ and hence $\|T_{zw}\|_\infty \leq \|T_{z^\epsilon w^\epsilon}\|_\infty < \gamma$. □

Lemma 3.2.2 *Given the plant $G(s)$ in (3.1) and a controller $K(s)$ which is γ-admissible for $G(s)$, then there exists an $\epsilon^*(K) > 0$ such that for all $\epsilon \in (0, \epsilon^*(K))$, $K(s)$ is γ-admissible for $G^\epsilon(s)$.*

Proof: Suppose a controller has been implemented which ensures $\|T_{zw}\|_\infty < \gamma$. Clearly, if $\epsilon = 0$ then $\|T_{z^\epsilon w^\epsilon}\|_\infty = \|T_{zw}\|_\infty$. Since the singular values of $T_{z^\epsilon w^\epsilon}(j\omega)$ vary continuously with ϵ (see [67]), there will be some $\epsilon^*(K) > 0$ such that $\epsilon \in (0, \epsilon^*(K))$ implies $\|T_{z^\epsilon w^\epsilon}\|_\infty < \gamma$. □

Theorem 3.2.1 *Given the plant $G(s)$ in (3.1), there exists a γ-admissible controller for $G(s)$ if and only if there exists an $\epsilon^* > 0$ such that for all $\epsilon \in (0, \epsilon^*)$, there exists a γ-admissible controller for $G^\epsilon(s)$.*

Proof: $\Rightarrow$ Let there exist a γ-admissible controller $K(s)$ for $G(s)$. From Lemma 3.2.2, there exists an $\epsilon^* = \epsilon^*(K)$ such that there exists a γ-admissible controller for $G^\epsilon(s)$ whenever $\epsilon \in (0, \epsilon^*)$.

$\Leftarrow$ Suppose $\exists\ \epsilon^* > 0$ such that there exists a γ-admissible controller for $G^\epsilon(s)$, $\forall\ \epsilon \in (0, \epsilon^*)$. From Lemma 3.2.1 it follows that $K^\epsilon(s)$ is a γ-admissible controller for $G(s)$, $\forall\ \epsilon \in (0, \epsilon^*)$. □

3.3 Controller Existence Conditions.

It is shown in this section that state-space existence results for the standard $\mathcal{H}_\infty$ problem lead to similar state-space results for the doubly nonstandard case. This connection is made via the family of augmented plants introduced in the previous section. Under the hypothesis that each of the standard assumptions holds for the realization (3.13) of $G^\epsilon(s)$, standard state-space existence conditions for γ-admissible controllers are first derived. It is then shown how the augmentation matrices $\bar{B}_1$ and $\bar{C}_1$ in the realization (3.13) of $G^\epsilon(s)$ can be chosen such that each standard assumption indeed holds. Subsequently, a limiting argument applied to the ϵ-dependent existence conditions establishes

ϵ-independent existence conditions for the nonstandard problem.

3.3.1 ϵ-Dependent Existence Conditions.

Roughly speaking, one conclusion of the previous section is that the existence of a γ-admissible controller for a nonstandard plant is *equivalent* to the existence of a γ-admissible controller for a (standard) ϵ-augmented plant for small enough ϵ. In this subsection, we investigate the consequence of this observation in state-space terms.

We first introduce the following algebraic Riccati equations which arise when the standard state space $\mathcal{H}_\infty$ theory summarized in Lemma 2.3.1 of Chapter 2 is applied to $G^\epsilon(s)$ with $\epsilon > 0$:

$$\begin{aligned} 0 &= X_\epsilon A_{ZX} + A_{ZX}^T X_\epsilon + X_\epsilon Q(\epsilon) X_\epsilon, && (3.21)\\ 0 &= Y_\epsilon A_{ZY}^T + A_{ZY} Y_\epsilon + Y_\epsilon P(\epsilon) Y_\epsilon, && (3.22)\end{aligned}$$

with the definitions:

$$\begin{aligned} A_{ZX} &= (A - B_2 D_{12}{}^{\dagger} C_1 - B_2 D_{12}{}^{\perp} \bar{C}_1), && (3.23)\\ A_{ZY} &= (A - B_1 D_{21}{}^{\dagger} C_2 - \bar{B}_1 D_{21}{}^{\perp} C_2), && (3.24)\\ Q(\epsilon) &= \gamma^{-2} B_1 B_1^T + \gamma^{-2}\epsilon^2 \bar{B}_1 \bar{B}_1^T - B_2 D_{12}{}^{\dagger} (D_{12}{}^{\dagger})^T B_2^T \\ &\quad -\frac{1}{\epsilon^2} B_2 D_{12}{}^{\perp} (D_{12}{}^{\perp})^T B_2^T, && (3.25)\\ P(\epsilon) &= \gamma^{-2} C_1^T C_1 + \gamma^{-2}\epsilon^2 \bar{C}_1^T \bar{C}_1 - C_2^T (D_{21}{}^{\dagger})^T D_{21}{}^{\dagger} C_2 \\ &\quad -\frac{1}{\epsilon^2} C_2^T (D_{21}{}^{\perp})^T D_{21}{}^{\perp} C_2. && (3.26)\end{aligned}$$

A solution X_ϵ of a Riccati equation of the form (3.21) is said to be *stabilizing* if the matrix $A_{ZX} + Q(\epsilon)X_\epsilon$ is stable (has all eigenvalues in the open left half-plane), whilst a solution Y_ϵ of an ARE of the form (3.22) is said to be stabilizing if $(A_{ZY} + Y_\epsilon P(\epsilon))$ is stable.

Lemma 3.3.1 *Let $G(s)$ be a doubly-nonstandard plant realized as in (3.1), satisfying* **A.1** , **A.2** *and* **A.3** *. Suppose $\bar{B}_1$ and $\bar{C}_1$ are chosen such that for all $\epsilon > 0$, $G^\epsilon(s)$, as realized in (3.13), satisfies* **A.3**.

1. *When they exist, nonnegative definite stabilizing solutions of (3.21) and (3.22) are unique.*

2. *There exists a γ-admissible controller for a doubly nonstandard plant $G(s)$ if and only if there exists an $\epsilon^* > 0$ such that for all $\epsilon \in (0, \epsilon^*)$, the algebraic Riccati equations (3.21) and (3.22) have nonnegative definite stabilizing solutions X_ϵ and Y_ϵ satisfying $\rho(X_\epsilon Y_\epsilon) < \gamma^2$.*

Proof: A proof is presented in Appendix B.1. □

Whilst Lemma 3.3.1 does give necessary and sufficient conditions for the existence of nonstandard controllers, it is of limited immediate use. Firstly, the result depends on $\bar{B}_1$ and $\bar{C}_1$ having been chosen such that $G^\epsilon(s)$satisfies **A.3**. Secondly, the dependence in the Riccati equations on ϵ is problematic since we have no knowledge in general of the size of ϵ^*. Thirdly, direct implementation of an ϵ-augmented controller is likely to be subject to numerical difficulties if ϵ^* is very small since the Riccati equations have entries which are unbounded as $\epsilon \to 0$; for example one cannot eliminate the ϵ-dependence of the Riccati equation (3.21) by the limiting process $\epsilon \to 0$ since the term $-\frac{1}{\epsilon^2}B_2{D_{12}}^\perp({D_{12}}^\perp)^T B_2^T$ in the Riccati equation diverges.

It is demonstrated subsequently that a careful construction of $\bar{C}_1$ and $\bar{B}_1$ guarantees not only that **A.3** holds for $G^\epsilon(s)$, but also that limiting solutions X_0 and Y_0 of the ϵ-dependent AREs exist. The next section contains important observations on the structure of nonstandard generalized plants which lead to this construction.

3.3.2 Choice of Matrices for Augmentation.

In this subsection, it is shown how the observations on the invariant zeros of $G_{12}(s)$ and/or $G_{21}(s)$ and on the associated canonical forms which were made in section 3.1 suggest a choice of augmentation which ensures that $G^\epsilon(s)$satisfies each of the assumptions **A.1** , **A.2** , **A.3** and **A.4**. Moreover, these results facilitate a simplification in the structure of the two ϵ-dependent Riccati equations (3.21) and (3.22). The following lemma is the key result in this subsection.

Lemma 3.3.2

1. *Let T be a similarity transformation associated with the controllability canonical form in (3.8) and (3.9). Given the selection*

$$\bar{C}_1 = -\begin{pmatrix} L_1 & L_2 \end{pmatrix} T^{-1}, \tag{3.27}$$

where L_2 is any matrix such that $A_1 + \beta_F L_2$ is stable and L_1 is a free matrix, the nonnegative definite stabilizing solution to the ϵ-dependent Riccati equation (3.21), when it exists, is independent of the matrices L_1 and L_2 (provided L_2 is stabilizing in the sense described above) and has the form:

$$X_\epsilon = (T^T)^{-1}\begin{pmatrix} \Psi_\epsilon & 0 \\ 0 & 0 \end{pmatrix} T^{-1}, \tag{3.28}$$

where Ψ_ϵ is a square matrix having the same dimensions as A_0. In addition, X_ϵ satisfies the following equality:

$$X_\epsilon B_2 {D_{12}}^\perp = 0. \tag{3.29}$$

2. *Let U be a similarity transformation associated with the observability canonical form in (3.11) and (3.12). Given the selection*

$$\bar{B}_1 = -U^{-1}\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}, \tag{3.30}$$

where M_2 is any matrix such that $\alpha_1 + M_2\beta_H$ is stable and M_1 is a free matrix, the nonnegative definite stabilizing solution to the ϵ-dependent Riccati equation (3.22), when it exists, is independent of the matrices M_1 and M_2 (provided M_2 is stabilizing in the sense described above) and has the form:

$$Y_\epsilon = U^{-1}\begin{pmatrix} \Theta_\epsilon & 0 \\ 0 & 0 \end{pmatrix}\left(U^T\right)^{-1}, \tag{3.31}$$

where Θ_ϵ is a square matrix having the same dimensions as α_0. In addition, Y_ϵ satisfies the following equality:

$$D_{21}{}^{\perp}C_2Y_\epsilon = 0. \tag{3.32}$$

3. *With $\bar{B}_1$ and $\bar{C}_1$ chosen according to items 1 and 2 respectively, the partitions $G^\epsilon_{12}(s)$ and $G^\epsilon_{21}(s)$ of $G^\epsilon(s)$as defined in (3.13) inherit the closed-right half plane invariant zeros of $G_{12}(s)$ and $G_{21}(s)$ respectively, but do not acquire any additional closed right half plane invariant zeros. Hence, under these conditions, $G^\epsilon(s)$as defined in (3.13) satisfies assumption* **A.3**.

Proof:
1. With definitions $\tilde{X}_\epsilon = T^TX_\epsilon T$ and $\tilde{Q}(\epsilon) = T^{-1}Q(\epsilon)(T^{-1})^T$, we return to the Riccati equation (3.21), expressed in the basis corresponding to the transformation T:

$$\begin{aligned} 0 = \tilde{X}_\epsilon &\left(\begin{pmatrix} A_0 & 0 \\ A_{01} & A_1 \end{pmatrix} - \begin{pmatrix} 0 \\ \beta_F \end{pmatrix}\bar{C}_1T\right) \\ &+ \left(\begin{pmatrix} A_0 & 0 \\ A_{01} & A_1 \end{pmatrix} - \begin{pmatrix} 0 \\ \beta_F \end{pmatrix}\bar{C}_1T\right)^T\tilde{X}_\epsilon + \tilde{X}_\epsilon\tilde{Q}(\epsilon)\tilde{X}_\epsilon. \end{aligned} \tag{3.33}$$

If we choose a matrix $\bar{C}_1$ which stabilizes the controllable modes corresponding to A_1, simplification of the Riccati equation follows from the following argument; since the pair (A_1, β_F) is controllable, it is possible to find an L_2 with $(A_1 + \beta_FL_2)$ stable and hence a $\bar{C}_1$ as described in item *1* exists. With such a choice of $\bar{C}_1$, the Riccati equation, transformed as in equation (3.33), can be expressed thus:

$$\begin{aligned} 0 = \tilde{X}_\epsilon &\begin{pmatrix} A_0 & 0 \\ A_{01} + \beta_FL_1 & A_1 + \beta_FL_2 \end{pmatrix} \\ &+ \left(\begin{pmatrix} A_0 & 0 \\ A_{01} + \beta_FL_1 & A_1 + \beta_FL_2 \end{pmatrix} + \tilde{Q}(\epsilon)\tilde{X}_\epsilon\right)^T\tilde{X}_\epsilon. \end{aligned} \tag{3.34}$$

If one right-multiplies this equation by the matrix $\begin{pmatrix} 0 \\ I \end{pmatrix}$ (where the identity matrix has the same dimensions as A_1), one obtains:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \tilde{X}_\epsilon \begin{pmatrix} 0 \\ I \end{pmatrix} (A_1 + \beta_F L_2) + \left(\begin{pmatrix} A_0 & 0 \\ A_{01} + \beta_F L_1 & A_1 + \beta_F L_2 \end{pmatrix} + \tilde{Q}(\epsilon)\tilde{X}_\epsilon \right)^T \tilde{X}_\epsilon \begin{pmatrix} 0 \\ I \end{pmatrix} \qquad (3.35)$$

Note first that $(A_1 + \beta_F L_2)$ has been designed stable and that

$$\left(\begin{pmatrix} A_0 & 0 \\ A_{01} + \beta_F L_1 & A_1 + \beta_F L_2 \end{pmatrix} + \tilde{Q}(\epsilon)\tilde{X}_\epsilon \right)$$

is stable since X_ϵ is by hypothesis a stabilizing solution of (3.21). The stability of these two matrices allows us to deduce from (3.35) and the well known Lemma of Lyapunov (see Lemma 0.0.1 in the summary of **Notation, Definitions and Fundamental Results** at the beginning of this thesis) to conclude that $\tilde{X}_\epsilon \begin{pmatrix} 0 \\ I \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and hence that the $(1,2)$ and $(2,2)$ blocks of $\tilde{X}_\epsilon$ are zero matrices. Since $\tilde{X}_\epsilon$ is symmetric, its $(2,1)$ block is also a zero matrix. This yields the structure of X_ϵ shown in equation (3.28).

Let the nonzero (1,1) partition of $\tilde{X}_\epsilon$ be Ψ_ϵ. From examination of the equation (3.34), we see that Ψ_ϵ satisfies

$$\Psi_\epsilon A_0 + A_0^T \Psi_\epsilon + \Psi_\epsilon \begin{pmatrix} I & 0 \end{pmatrix} \tilde{Q}(\epsilon) \begin{pmatrix} I & 0 \end{pmatrix}^T \Psi_\epsilon = 0 \qquad (3.36)$$

from which it is clear that Ψ_ϵ is independent of $\bar{C}_1$[1]. Since T is also independent of $\bar{C}_1$, we deduce from (3.28) that X_ϵ is also. The identity (3.29) follows directly from the structure of X_ϵ exhibited in (3.28) and the structure of $B_2 D_{12}{}^{\perp}$ in (3.9).

2. The proof is analogous to that for part *1*.

3. Application of the argument used in Appendix A to $G_{12}^\epsilon(s)$ (with the choice of $\bar{C}_1$ given in part *1*) and $G_{21}^\epsilon(s)$ (with the choice of $\bar{B}_1$ given in part *2*), yields the result. □

3.3.3 ϵ-Independent Existence Conditions.

Recall that the special choice of augmentation in Lemma 3.3.2 makes application of Lemma 3.3.1 to G^ϵ possible. This results in existence conditions for the nonstandard plant in terms of the solutions of the two ϵ-dependent Riccati equations (3.21) and (3.22). In this subsection, we first demonstrate how the special choice of plant augmentation matrices also leads to the elimination of the divergent terms in the ϵ-dependent Riccati equations. It is possible to express controller existence conditions for $G^\epsilon(s)$ in terms of modified ϵ-dependent Riccati equations which have no terms which diverge as $\epsilon \to 0$. The limit as $\epsilon \to 0$ of the modified ϵ-dependent AREs therefore exist and are themselves

[1]Note that it is still dependent on $\bar{B}_1$ which is present in $Q(\epsilon)$, however this dependence disappears in the limit as $\epsilon \to 0$.

AREs. Solutions X_0 and Y_0 of these ϵ-independent AREs are then shown to exist if and only if there exist solutions of the family of modified ϵ-dependent AREs. This result is established using standard analysis results, including the implicit function theorem. The conclusion is that existence of a nonstandard γ-admissible controller is equivalent to the existence of nonnegative definite stabilizing solutions to the ϵ-independent AREs and satisfaction of a coupling condition $\rho(X_0Y_0) < \gamma^2$.

The Modified ϵ-Dependent AREs.

Consider the modified pair of ϵ-dependent algebraic Riccati equations:

$$\mathcal{X}_\epsilon A_{ZX} + A_{ZX}^T\mathcal{X}_\epsilon + \mathcal{X}_\epsilon\mathcal{Q}(\epsilon)\mathcal{X}_\epsilon = 0, \tag{3.37}$$
$$\mathcal{Y}_\epsilon A_{ZY}^T + A_{ZY}\mathcal{Y}_\epsilon + \mathcal{Y}_\epsilon\mathcal{P}(\epsilon)\mathcal{Y}_\epsilon = 0, \tag{3.38}$$

with the definitions (compare with (3.25) and (3.26))

$$\mathcal{Q}(\epsilon) = \gamma^{-2}B_1B_1^T + \gamma^{-2}\epsilon^2\bar{B}_1\bar{B}_1^T - B_2D_{12}{}^{\dagger}(D_{12}{}^{\dagger})^TB_2^T \tag{3.39}$$
$$\mathcal{P}(\epsilon) = \gamma^{-2}C_1^TC_1 + \gamma^{-2}\epsilon^2\bar{C}_1^T\bar{C}_1 - C_2^T(D_{21}{}^{\dagger})^TD_{21}{}^{\dagger}C_2 \tag{3.40}$$

where A_{ZX} and A_{ZY} are defined in (3.23) and (3.24) respectively. The important difference between these equations and the ϵ-dependent AREs for X_ϵ and Y_ϵ is that the quadratic terms $\mathcal{Q}(\epsilon)$ and $\mathcal{P}(\epsilon)$ do *not* diverge as $\epsilon \to 0$. The next lemma claims that for the purposes of checking for controller existence, one can replace equations (3.21) and (3.22) with (3.37) and (3.38) respectively.

Lemma 3.3.3 *Let $\bar{B}_1$ and $\bar{C}_1$ be chosen according to Lemma 3.3.2.*

1. *When they exist, nonnegative definite stabilizing solutions $\mathcal{X}_\epsilon$ and $\mathcal{Y}_\epsilon$ to (3.37) and (3.38) are unique.*

2. *X_ϵ and Y_ϵ are nonnegative definite stabilizing solutions of (3.21) and (3.22) if and only if $\mathcal{X}_\epsilon = X_\epsilon$ and $\mathcal{Y}_\epsilon = Y_\epsilon$ are nonnegative definite stabilizing solutions of (3.37) and (3.38).*

Proof: A proof of this result is presented in Appendix B.2. □

Limiting Results for a Family of AREs.

The next lemma is needed to prove the ϵ-independent existence result in Theorem 3.3.1 which concludes this section.

Lemma 3.3.4 *Given real matrices $A \in \mathbb{R}^{n\times n}$ and $Q = Q^T \in \mathbb{R}^{n\times n}$, suppose the algebraic Riccati equation*

$$XA + A^TX + XQX = 0 \tag{3.41}$$

has a nonnegative definite stabilizing solution $X = X_0$, where Q is symmetric. Let $Q(\eta) : \mathbb{R} \to \mathbb{R}^{n\times n}$ be a continuously differentiable symmetric function of $\eta \in \mathbb{R}$ which satisfies $Q(0) = Q$. For each $\eta \in \mathbb{R}$, consider the algebraic Riccati equation:

$$X_\eta A + A^T X_\eta + X_\eta Q(\eta) X_\eta = 0. \tag{3.42}$$

1. *There exists an $\eta_1 > 0$ such that for all $\eta \in (-\eta_1, \eta_1)$, there exists a stabilizing solution X_η of (3.42), moreover in that interval;*
   a) X_η is a continuous function of η.
   b) The derivative $\frac{dX_\eta}{d\eta}$ exists and varies continuously with η.
   c) X_η is symmetric.
2. *In addition, suppose there exists an $\hat{\eta} > 0$ s.t. for all $\eta \in (0, \hat{\eta})$, $\frac{dQ}{d\eta} \geq 0$. Then with $\eta_2 = \min\{\hat{\eta}, \eta_1\} > 0$ it follows that $X_\eta \geq X_0 \geq 0$, for all $\eta \in (0, \eta_2)$.*

Proof: See Appendix C.1 for a simple proof from first principles, based on application of the implicit function theorem to the Riccati equation (3.42). Related continuity results have been obtained in [88] for algebraic Riccati equations where $Q(\eta)$ is nonnegative-definite, a condition which *cannot* be guaranteed here. □

Lemma 3.3.5 *Adopting the notation of Lemma 3.3.4, assume that $\frac{dQ}{d\eta} \geq 0$ for all $\eta > 0$. Suppose also that there exists an $\eta^* > 0$ such that for all $\eta \in (0, \eta^*)$, there exists a nonnegative definite stabilizing solution X_η of (3.42), then there exists a nonnegative definite stabilizing solution $X = X_0$ of (3.41). Moreover, $X_\eta \geq X_0$ for all $\eta \in (0, \eta^*)$.*

Proof: Refer to Appendix C.2. □

Corollary 3.3.1 *Adopting the notation of Lemma 3.3.4, assume that $\frac{dQ}{d\eta} \geq 0$, for all $\eta \geq 0$. The equation (3.41) has a nonnegative definite stabilizing solution $X = X_0$ if and only if there exists an $\eta^* > 0$ such that for all $\eta \in (0, \eta^*)$, (3.42) has a nonnegative definite stabilizing solution X_η. Moreover, $X_\eta \geq X_0$ for all $\eta \in (0, \eta^*)$.*

Proof: $\Rightarrow$ Let η_1 be defined by application of part *1* of Lemma 3.3.4 to (3.41). Since by hypothesis $\frac{dQ}{d\eta} \geq 0$, $\forall\ \eta \geq 0$, define $\eta_2 > 0$ by application of part *2* of Lemma 3.3.4 to (3.41). Choose $\eta^* = \min(\eta_1, \eta_2)$. The *if* part of the desired result then follows. $\forall\ \eta \in (0, \eta^*)$.
$\Leftarrow$ Follows from Lemma 3.3.5 as does the fact that $X_\eta \geq X_0$, $\forall\ \eta \in (0, \eta^*)$. □

Proof of the Doubly Nonstandard Existence Result.

The following theorem is one of the main results of this chapter. It presents necessary and sufficient conditions under which there exist γ-admissible controllers for the doubly nonstandard generalized plant. It is shown in the next section that once the conditions of this theorem are satisfied, it is possible to construct all doubly nonstandard $\mathcal{H}_\infty$ controllers.

Theorem 3.3.1 *Let $G(s)$ be a doubly nonstandard plant realized as in (3.1), satisfying* **A.1** , **A.2** *and* **A.3** . *Suppose $\bar{B}_1$ and $\bar{C}_1$ are constructed in the manner described in Lemma 3.3.2 via matrices L_1, L_2, M_1 and M_2, such that the matrices $A_1 + \beta_F L_2$ and $\alpha_1 + M_2\beta_H$ are stable. Let A_{ZX} and A_{ZY} be given by (3.23) and (3.24) respectively.*

A necessary and sufficient condition for the existence of a γ-admissible controller for $G(s)$ is that the following conditions hold:

1. There exists a stabilizing solution $X_0 \geq 0$ of the ARE

$$X_0 A_{ZX} + A_{ZX}{}^T X_0 + X_0(\gamma^{-2} B_1 B_1^T - B_2 D_{12}{}^{\dagger}(D_{12}{}^{\dagger})^T B_2^T) X_0 = 0. \tag{3.43}$$

2. There exists a stabilizing solution $Y_0 \geq 0$ of the ARE

$$Y_0 A_{ZY}{}^T + A_{ZY} Y_0 + Y_0(\gamma^{-2} C_1^T C_1 - C_2^T (D_{21}{}^{\dagger})^T D_{21}{}^{\dagger} C_2) Y_0 = 0. \tag{3.44}$$

3.

$$\rho(X_0 Y_0) < \gamma^2. \tag{3.45}$$

When such matrices X_0 and Y_0 exist, they are unique and independent of the particular choice of L_1, L_2, M_1 and M_2 and thereby of $\bar{B}_1$ and $\bar{C}_1$ as described in Lemma 3.3.2, provided the matrices L_2 and M_2 are chosen to stabilize $A_1 + \beta_F L_2$ and $\alpha_1 + M_2\beta_H$, respectively.

Proof:

Necessity: Assume a γ-admissible control law for $G(s)$ has been found. By Lemma 3.3.1 in conjunction with Lemma 3.3.3, it follows that the Riccati equations (3.37) and (3.38) have nonnegative definite stabilizing solutions $\mathcal{X}_\epsilon$ and $\mathcal{Y}_\epsilon$ for some finite ϵ-interval $(0, \epsilon^*)$.

Observe from (3.39) that $\frac{d\mathcal{Q}}{d\epsilon} = \gamma^{-2} 2\epsilon \bar{B}_1 \bar{B}_1^T \geq 0$, $\forall\ \epsilon \geq 0$. Corollary 3.3.1 can thus be applied to (3.37) with $\eta = \epsilon$, $A = A_{ZX}$, $Q(\eta) = \mathcal{Q}(\epsilon)$ and $X_\eta = \mathcal{X}_\epsilon$, thus establishing the existence of a nonnegative definite solution X_0 to (3.43). An analogous argument establishes the existence of a nonnegative definite solution Y_0 of (3.44) from the equation (3.38) for $\mathcal{Y}_\epsilon$.

To establish the coupling condition $\rho(X_0 Y_0) < \gamma^2$, observe first that $\mathcal{X}_\epsilon \geq X_0$ and $\mathcal{Y}_\epsilon \geq Y_0$ for any $\epsilon \in (0, \epsilon^*)$, facts which also follow from Corollary 3.3.1. Next note the following chain of inequalities: $\gamma^2 > \rho(\mathcal{X}_\epsilon \mathcal{Y}_\epsilon) = \rho(\mathcal{Y}_\epsilon^{\frac{1}{2}} \mathcal{X}_\epsilon \mathcal{Y}_\epsilon^{\frac{1}{2}}) \geq \rho(\mathcal{Y}_\epsilon^{\frac{1}{2}} X_0 \mathcal{Y}_\epsilon^{\frac{1}{2}}) = \rho(X_0^{\frac{1}{2}} \mathcal{Y}_\epsilon X_0^{\frac{1}{2}}) \geq \rho(X_0^{\frac{1}{2}} Y_0 X_0^{\frac{1}{2}}) = \rho(X_0 Y_0)$

Sufficiency: Suppose one has nonnegative definite stabilizing solutions of both (3.43) and (3.44) which satisfy (3.45). We now aim to prove the existence of a γ-admissible controller by establishing the equivalent conditions described in Lemma 3.3.1.

Let X_0 be the nonnegative definite stabilizing solution to (3.43). By Corollary 3.3.1 it follows that $\exists\ \hat{\epsilon} > 0$ such that $\forall\ \epsilon \in (0, \hat{\epsilon}) \Rightarrow \exists\ \mathcal{X}_\epsilon$ which is a nonnegative stabilizing solution of (3.37). Analogous arguments establish the existence of an $\bar{\epsilon} > 0$ such that $\forall\ \epsilon \in (0, \bar{\epsilon}) \Rightarrow \exists\ \mathcal{Y}_\epsilon$ which is a nonnegative definite stabilizing solution of (3.38).

Note from Lemma 3.3.4 that $\mathcal{X}_\epsilon$ and $\mathcal{Y}_\epsilon$ vary continuously with ϵ since $\mathcal{Q}(\epsilon)$ and $\mathcal{P}(\epsilon)$ are continuously differentiable functions of ϵ. By hypothesis, $\rho(X_0 Y_0) < \gamma^2$. Since $\mathcal{X}_\epsilon$ and $\mathcal{Y}_\epsilon$ depend continuously on ϵ, so will the eigenvalues of their product and thus its spectral radius $\rho(\mathcal{X}_\epsilon \mathcal{Y}_\epsilon)$. Hence $\exists\ \epsilon_\rho > 0$ such that $\epsilon \in (0, \epsilon_\rho) \Rightarrow \rho(\mathcal{X}_\epsilon \mathcal{Y}_\epsilon) < \gamma^2$.

Let $\epsilon^* = \min(\hat{\epsilon}, \bar{\epsilon}, \epsilon_\rho)$. It follows from Lemma 3.3.3 and the above argument that provided $\epsilon \in (0, \epsilon^*)$, $X_\epsilon = \mathcal{X}_\epsilon$ and $Y_\epsilon = \mathcal{Y}_\epsilon$ are nonnegative definite stabilizing solutions of (3.21) and (3.22) and $\rho(X_\epsilon Y_\epsilon) < \gamma^2$. One can then apply Lemma 3.3.1 to establish the existence of a γ-admissible controller.

An argument similar to that applied to (3.21) and (3.22) in the proof of Lemma 3.3.2 can be applied to the Riccati equations (3.43) and (3.44) with the same basis transformations T and U, to reveal that both X_0 and Y_0 are independent of $\bar{B}_1$ and $\bar{C}_1$.
□

Remark: Note from the above proof that it follows that, provided $\bar{C}_1$ and $\bar{B}_1$ are chosen according to Lemma 3.3.2, limiting solutions of the ϵ-dependent equations (3.21) and (3.22) exist;

- $\lim_{\epsilon \to 0} X_\epsilon = X_0 \geq 0$
- $\lim_{\epsilon \to 0} Y_\epsilon = Y_0 \geq 0$

It can also be shown that the following facts hold:

- $\lim_{\epsilon \to 0}(A_{ZX} + Q(\epsilon) X_\epsilon)$ exists and has all eigenvalues in the open left half-plane.
- $\lim_{\epsilon \to 0}(A_{ZY} + P(\epsilon) Y_\epsilon)$ exists and has all eigenvalues in the open left half-plane.

Note that the above limits exist, despite the fact that entries in the Riccati equations diverge as $\epsilon \to 0$. This is due to the special structure of X_ϵ and Y_ϵ which was elucidated in Lemma 3.3.2. □

3.4 Controller Parametrization for the Augmented Plant.

This section, together with the next comprise a derivation of a parametrization of $\mathcal{K}^\gamma$, the set of *all* γ-admissible controllers for $G(s)$. We assume the existence of at least one γ-admissible controller for the doubly nonstandard generalized plant $G(s)$. The present section describes the first step in this process, whereby $\mathcal{K}_\epsilon^\gamma$, the set of all γ-admissible controllers for the ϵ-augmented plant $G^\epsilon(s)$, is investigated. (Henceforth, for the sake of brevity, any $K \in \mathcal{K}_\epsilon^\gamma$ is called an *ϵ-controller.*) The study of $\mathcal{K}_\epsilon^\gamma$ is motivated by the following reasoning: recall from Lemma 3.2.2 that all nonstandard controllers are expressible as an ϵ-controller provided ϵ is small enough. It is also true that all ϵ-controllers are γ-admissible for the nonstandard plant (see Lemma 3.2.1). Thus we can begin to build a state-space picture of $\mathcal{K}^\gamma$ by applying the results of the standard $\mathcal{H}_\infty$ theory to ϵ-augmented systems. This gives rise at first to a state-space description of $\mathcal{K}_\epsilon^\gamma$, the set of all ϵ-controllers. Subsequently, in section 3.5 we establish the controller form for all doubly nonstandard plants via limiting arguments involving $\mathcal{K}_\epsilon^\gamma$.

3.4.1 State-space Parametrization of all ϵ-Controllers.

By hypothesis, there exists at least one γ-admissible controller for $G(s)$. We assume that the matrices $\bar{B}_1$ and $\bar{C}_1$ which appear in the state space description of the generalized plant $G^\epsilon(s)$given in (3.13) are chosen according to Lemma 3.3.2. Recall that it was shown in section 3.3 that when $G^\epsilon(s)$is constructed using this augmentation, it satisfies each of the standard assumptions **A.1**, **A.2**, **A.3** and **A.4**. Lemma 3.3.1 then guarantees the existence of an ϵ^* such that for all $\epsilon \in (0, \epsilon^*)$, there exist stabilizing solutions X_ϵ and Y_ϵ of the ϵ-dependent AREs (3.21) and (3.22) which satisfy $\rho(X_\epsilon Y_\epsilon) < \gamma^2$.

Application of the Standard $\mathcal{H}_\infty$ Controller Formulae to $G^\epsilon(s)$.

Application of the standard controller formulae in Lemma 2.3.1 of Chapter 2 to the ϵ-augmented plant $G^\epsilon(s)$in (3.13) results in the following description of all ϵ-controllers:

$$K^\epsilon(s) = LFT\{M^\epsilon(s), N^\epsilon(s)\}. \tag{3.46}$$

In the above expression,

$$M^\epsilon(s) = \left(\begin{array}{c|ccc} \hat{A}^\epsilon + B_2 F_\infty^\epsilon + H_\infty^\epsilon \hat{C}_2^\epsilon & -H_\infty^\epsilon & Z_\epsilon \hat{B}_2^\epsilon D_{12}{}^\dagger & \epsilon^{-1} Z_\epsilon \hat{B}_2^\epsilon D_{12}{}^\perp \\ \hline F_\infty^\epsilon & 0 & D_{12}{}^\dagger & \epsilon^{-1} D_{12}{}^\perp \\ -D_{21}{}^\dagger \hat{C}_2^\epsilon & D_{21}{}^\dagger & 0 & 0 \\ -\epsilon^{-1} D_{21}{}^\perp \hat{C}_2^\epsilon & \epsilon^{-1} D_{21}{}^\perp & 0 & 0 \end{array}\right), \tag{3.47}$$

and N^ϵ is a free parameter having block partitioning compatible with that of the coefficient matrix M^ϵ of the linear fractional map above and with the property that

$$N^\epsilon = \begin{pmatrix} N_{11}^\epsilon & N_{12}^\epsilon \\ N_{21}^\epsilon & N_{22}^\epsilon \end{pmatrix} \in \mathcal{BH}_\infty^\gamma. \tag{3.48}$$

Recall that $\mathcal{BH}_\infty^\gamma$ is the set of all $M(s) \in \mathcal{RH}_\infty$ such that $\|M(s)\|_\infty < \gamma$.

Note here, with reference to Lemma 2.3.1 in Chapter 2 that we have chosen $V_{12}^\epsilon = D_{12}^\epsilon{}^{-1}$ and $V_{21}^\epsilon = D_{21}^\epsilon{}^{-1}$ as given in (3.18) and (3.19) respectively. Recall from (3.29) and (3.32) that $(D_{12}{}^\perp)^T B_2^T X_\epsilon = 0$ and $Y_\epsilon C_2^T (D_{21}{}^\perp)^T = 0$. These identities, together with the standard state space controller formulae in Lemma 2.3.1 lead to the expressions:

$$\begin{aligned}
F_\infty^\epsilon &= -D_{12}{}^\dagger C_1 - D_{12}{}^\perp \bar{C}_1 - D_{12}{}^\dagger (D_{12}{}^\dagger)^T B_2^T X_\epsilon, && (3.49)\\
H_\infty^\epsilon &= Z_\epsilon \left(-B_1 D_{21}{}^\dagger - \bar{B}_1 D_{21}{}^\perp - Y_\epsilon C_2^T (D_{21}{}^\dagger)^T D_{21}{}^\dagger\right), && (3.50)\\
\hat{A}^\epsilon &= A + \gamma^{-2} B_1 B_1^T X_\epsilon + \gamma^{-2}\epsilon^2 \bar{B}_1 \bar{B}_1^T X_\epsilon, && (3.51)\\
Z_\epsilon &= (I - \gamma^{-2} Y_\epsilon X_\epsilon)^{-1}, && (3.52)\\
\hat{C}_2^\epsilon &= C_2 + \gamma^{-2} D_{21} B_1^T X_\epsilon + \epsilon^2 \gamma^{-2} (D_{21}{}^\perp)^T \bar{B}_1^T X_\epsilon, && (3.53)\\
\hat{B}_2^\epsilon &= B_2 + \gamma^{-2} Y_\epsilon C_1^T D_{12} + \epsilon^2 \gamma^{-2} Y_\epsilon \bar{C}_1^T (D_{12}{}^\perp)^T. && (3.54)
\end{aligned}$$

Note that Z_ϵ is well-defined since $\rho(X_\epsilon Y_\epsilon) < \gamma^2$ for all $\epsilon \in (0, \epsilon^*)$.

Each controller of the form given in (3.46) is, by Lemma 3.2.1, also a doubly nonstandard controller. Note however, it is not true in general that *all* doubly nonstandard controllers are expressible in this form for a *fixed* $\epsilon > 0$. Thus (3.46) on its own cannot be used to describe *all* doubly nonstandard controllers. In addition, it suffers from numerical problems. Even if solutions of the Riccati equations are available, it is apparent from (3.47) that certain terms in M^ϵ diverge as $\epsilon \to 0$.

Alternative State Space description of all ϵ-Controllers.

It can be easily seen from (3.46), (3.47) and (3.48) that through a simple scaling procedure, the following alternative expression of all ϵ-controllers is available:

$$K^\epsilon(s) = LFT\left\{\tilde{M}^\epsilon, W^\epsilon\right\}, \tag{3.55}$$

with the free parameter

$$W^\epsilon = \begin{pmatrix} N_{11}^\epsilon & W_1^\epsilon \\ W_2^\epsilon & W_3^\epsilon \end{pmatrix} \in \mathcal{RH}_\infty, \tag{3.56}$$

which is subject to the constraint that

$$\begin{pmatrix} N_{11}^\epsilon & \epsilon W_1^\epsilon \\ \epsilon W_2^\epsilon & \epsilon^2 W_3^\epsilon \end{pmatrix} = N^\epsilon \in \mathcal{BH}_\infty^\gamma. \tag{3.57}$$

The new scaled coefficient matrix in the above description of all ϵ-controllers is given by:

$$\tilde{M}^\epsilon(s) = \left(\begin{array}{c|ccc} \hat{A}^\epsilon + B_2 F_\infty^\epsilon + H_\infty^\epsilon \hat{C}_2^\epsilon & -H_\infty^\epsilon & Z_\epsilon \hat{B}_2^\epsilon D_{12}{}^\dagger & Z_\epsilon \hat{B}_2^\epsilon D_{12}{}^\perp \\ \hline F_\infty^\epsilon & 0 & D_{12}{}^\dagger & D_{12}{}^\perp \\ -D_{21}{}^\dagger \hat{C}_2^\epsilon & D_{21}{}^\dagger & 0 & 0 \\ -D_{21}{}^\perp \hat{C}_2^\epsilon & D_{21}{}^\perp & 0 & 0 \end{array}\right) \tag{3.58}$$

with constituent matrices defined in (3.49) – (3.54). The above description of all ϵ-controllers does not suffer the same numerical difficulties as (3.46) and is the key to finding a complete description of doubly nonstandard controllers.

3.4.2 Limiting Behaviour of ϵ-Controllers.

The following lemma presents some properties of the alternative description of ϵ-controllers which was presented in the previous subsection.

Lemma 3.4.1 *Let $G(s)$ be a doubly nonstandard plant realized as in (3.1), satisfying* **A.1**, **A.2** *and* **A.3**. *Suppose $\bar{B}_1$ and $\bar{C}_1$ are chosen according to Lemma 3.3.2. Let A_{ZX} and A_{ZY} be given by (3.23) and (3.24) respectively. Assume that there exists a γ-admissible controller for $G(s)$ and let X_0 and Y_0 be the nonnegative definite stabilizing solutions of the AREs (3.43) and (3.44) respectively.*

Then the state-space realization of $\tilde{M}^\epsilon$ given in (3.58) varies continuously with ϵ and the following limit exists:

$$\lim_{\epsilon \to 0} \tilde{M}^\epsilon = \tilde{M}^0, \tag{3.59}$$

where

$$\tilde{M}^0 = \left(\begin{array}{c|ccc} \hat{A} + B_2 F_\infty + H_\infty \hat{C}_2 & -H_\infty & Z_0 \hat{B}_2 {D_{12}}^\dagger & B_2 {D_{12}}^\perp \\ \hline F_\infty & 0 & {D_{12}}^\dagger & {D_{12}}^\perp \\ -{D_{21}}^\dagger \hat{C}_2 & {D_{21}}^\dagger & 0 & 0 \\ -{D_{21}}^\perp C_2 & {D_{21}}^\perp & 0 & 0 \end{array} \right) \tag{3.60}$$

and

$$\begin{array}{rclrcl} F_\infty & = & -{D_{12}}^\dagger C_1 - {D_{12}}^\perp \bar{C}_1 - {D_{12}}^\dagger ({D_{12}}^\dagger)^T {B_2}^T X_0, & \hat{A} & = & A + \gamma^{-2} B_1 B_1^T X_0, \\ H_\infty & = & Z_0(-B_1 {D_{21}}^\dagger - \bar{B}_1 {D_{21}}^\perp - Y_0 C_2^T ({D_{21}}^\dagger)^T {D_{21}}^\dagger), & \hat{C}_2 & = & C_2 + \gamma^{-2} D_{21} B_1^T X_0, \\ Z_0 & = & (I - \gamma^{-2} Y_0 X_0)^{-1}, & \hat{B}_2 & = & B_2 + \gamma^{-2} Y_0 C_1^T D_{12}. \end{array}$$

Proof: Observe that the conditions assumed in the statement of this lemma ensure that those associated with Lemma 3.3.1 hold, which guarantees the existence of an $\epsilon^* > 0$ such that $\forall\ \epsilon \in (0, \epsilon^*)$, there exist nonnegative definite stabilizing solutions X_ϵ and Y_ϵ of the Riccati equations (3.21) and (3.22) which satisfy $\rho(X_\epsilon Y_\epsilon) < \gamma^2$. Recall also from Lemma 3.3.4, that X_ϵ, Y_ϵ and hence Z_ϵ vary continuously with ϵ in the given interval. From this fact and from inspection of the formulae (3.49) – (3.54), continuous dependence on ϵ of each element of the state-space realization of $\tilde{M}^\epsilon$ (3.58) can be deduced.

Recall from the previous section that limiting values X_0 and Y_0 of X_ϵ and Y_ϵ exist. The following limits result: $\lim_{\epsilon\to 0} \hat{A}^\epsilon = \hat{A}$, $\lim_{\epsilon\to 0} F_\infty^\epsilon = F_\infty$, $\lim_{\epsilon\to 0} \hat{C}_2^\epsilon = \hat{C}_2$ and $\lim_{\epsilon\to 0} \hat{B}_2^\epsilon = \hat{B}_2$, where formulae for these limits are given in the Lemma statement. Observe that since $\rho(X_0 Y_0) < \gamma^2$, $Z_0 = (I - \gamma^{-2} Y_0 X_0)^{-1}$ is well defined and note that $\lim_{\epsilon\to 0} Z_\epsilon = Z_0$. It then follows that $\lim_{\epsilon\to 0} H_\infty^\epsilon = H_\infty$.

The above facts establish the convergence of each block of (3.58) to the corresponding blocks of (3.60), except the blocks $(4,1)$ and $(1,4)$. The convergence of these blocks may be established by noting the following equalities:

$$\begin{aligned} Z_\epsilon \hat{B}_2^\epsilon {D_{12}}^\perp &= Z_\epsilon (B_2 {D_{12}}^\perp + \epsilon^2 \gamma^{-2} Y_\epsilon \bar{C}_1^T ({D_{12}}^\perp)^T {D_{12}}^\perp) \\ &= Z_\epsilon (I - \gamma^{-2} Y_\epsilon X_\epsilon + \gamma^{-2} Y_\epsilon X_\epsilon) B_2 {D_{12}}^\perp \end{aligned}$$

$$
\begin{aligned}
& +\epsilon^2\gamma^{-2}Z_\epsilon Y_\epsilon \bar{C}_1^T(D_{12}{}^\perp)^T D_{12}{}^\perp \\
&= B_2 D_{12}{}^\perp + \epsilon^2\gamma^{-2}Z_\epsilon Y_\epsilon \bar{C}_1^T(D_{12}{}^\perp)^T D_{12}{}^\perp, \quad (3.61)\\
-D_{21}{}^\perp \hat{C}_2^\epsilon &= -(D_{21}{}^\perp C_2 + \epsilon^2\gamma^{-2}D_{21}{}^\perp(D_{21}{}^\perp)^T\bar{B}_1^T X_\epsilon). \quad (3.62)
\end{aligned}
$$

In deriving (3.61), we have made use of the identity (3.29). □

Whilst the above Lemma demonstrates the existence of $\tilde{M}^0$ (the limit of $\tilde{M}^\epsilon$ as $\epsilon \to 0$ of the coefficient matrix of γ-admissible controllers for the ϵ-augmented generalized plant), it does not make clear how one can fully describe the freedom in $\mathcal{H}_\infty$ control laws; this is addressed in the next section. With regard to the freedom in this parametrization, the identity (3.57) gives some hint as to the likely form of the parametrization for nonstandard controllers. It follows from this expression that the limitations on the infinity norm of the transfer functions W_i^ϵ $(i = 1, 2, 3)$ reduce as ϵ becomes small, whilst these transfer function matrices must remain stable for finite ϵ. The limitation on the infinity norm of N_{11}^ϵ is maintained as $\epsilon \to 0$, however.

3.5 All $\mathcal{H}_\infty$ Controllers for the Nonstandard Plant.

The following theorem is the main controller parametrization result in this chapter. It presents a state-space construction of all γ-admissible controllers for the doubly nonstandard plant. The remainder of this section is concerned with a proof of this result.

Theorem 3.5.1 *Let $G(s)$ be a doubly nonstandard plant, realized as in (3.1), satisfying* **A.1**, **A.2** *and* **A.3**. *Let there exist at least one γ-admissible controller for $G(s)$. Let $\bar{B}_1$ and $\bar{C}_1$ be chosen in accordance with Lemma 3.3.2. Let $\tilde{M}^0$ be defined as in Lemma 3.4.1 and define the set*

$$
\mathcal{K}_0^\gamma = \left\{ K \mid K = LFT\{\tilde{M}^0, \begin{pmatrix} N & W_1 \\ W_2 & W_3 \end{pmatrix}\}\ ,\ \ N \in \mathcal{BH}_\infty^\gamma,\ W_i \in \mathcal{RH}_\infty,\ i = 1,2,3 \right\}. \quad (3.63)
$$

Then $\mathcal{K}_0^\gamma$ is the set of all γ-admissible nonstandard controllers,

$$
\mathcal{K}^\gamma = \mathcal{K}_0^\gamma.
$$

We set about proving Theorem 3.5.1 by showing the set inclusions $\mathcal{K}^\gamma \subset \mathcal{K}_0^\gamma$ and $\mathcal{K}_0^\gamma \subset \mathcal{K}^\gamma$ in Lemmas 3.5.3 and 3.5.4 respectively. Before proceeding, we present a number of results in the next subsection which are of utility in proving the main result.

3.5.1 Some Continuity Properties of Linear Fractional Transformations.

To facilitate the proof of Theorem 3.5.1, the following lemma describes some properties of a particular class of linear fractional maps.

Lemma 3.5.1

1. *Let $M(s)$ be a proper, real-rational transfer function matrix with partitioning*

$$M(s) = \begin{pmatrix} M_{11}(s) & M_{12}(s) \\ M_{21}(s) & M_{22}(s) \end{pmatrix}$$

and the following properties:

a) $M_{12}(s)$ and $M_{21}(s)$ have proper inverses.

b) M_{22} is strictly proper. (i.e. $\lim_{s\to\infty} M_{22}(s) = 0$.)

Then the linear fractional map

$$K = LFT\{M, N\} = M_{11} + M_{12}N(I - M_{22}N)^{-1}M_{21} \tag{3.64}$$

defines a one-to-one relationship between real rational proper transfer function matrices K and N. In other words, the mapping defined by (3.64) is invertible.

2. *Let $\psi \in \mathbb{R}^k$ be a vector of real parameters. Suppose that a family of state-space realizations of coefficient matrices M^ψ for the linear fractional map in (3.64) is given, which satisfies properties a) and b) in 1 for the considered range of values of the parameter ψ. Suppose also that one is given a family of state space realizations of either one of the real rational transfer function matrices $N^\psi(s)$ or $K^\psi(s)$ in (3.64). Suppose that each of the given realizations is continuously dependent on ψ. It follows that there exists a state-space realization of a third system defined by (3.64) (i.e. either $K^\psi(s)$ or $N^\psi(s)$) which depends continuously on ψ.*

3. *Suppose a transfer-function matrix $W^\psi(s)$ has a state-space realization with matrices continuously dependent on a parameter $\psi \in \mathbb{R}^k$ and suppose that $W^\psi(s) \in \mathcal{RH}_\infty$ for all values of the parameter of interest. Then $\|W^\psi\|_\infty$ is a continuous function of the parameter ψ.*

Proof:

1. Suppose first that a proper real-rational $N(s)$ is given. $(I - M_{22}N)^{-1}$ is well-defined and proper since M_{22} is strictly proper. Properness of M_{11}, M_{12} and M_{21} follow from properness of M. With $K(s)$ defined in (3.64), the above observations demonstrate that it is proper and real-rational.

Now suppose a proper real-rational $K(s)$ is given. With reference to (3.64), note that the quantity

$$N(I - M_{22}N)^{-1} = M_{12}^{-1}(K - M_{11})M_{21}^{-1} = V \tag{3.65}$$

exists due to the assumption *1 a)*. Note that the quantity $V = M_{12}^{-1}(K - M_{11})M_{21}^{-1}$ is proper and real-rational. This follows from the properness of the inverses M_{12}^{-1} and M_{21}^{-1} (which was assumed in item *1 a)*) and of both M_{11} and K. From (3.65), it follows that $(I - VM_{22})N = V$. Since V is proper and M_{22} is strictly proper, the inverse

$(I - VM_{22})^{-1}$ is well defined. It follows that the real-rational transfer function matrix $N = (I - VM_{22})^{-1}V$ is uniquely defined, proper and satisfies (3.64).

2. Since all variations in the parameter ψ are assumed to be such that the assumptions on the coefficient matrix $M(s)$ in item *1* are preserved for $M^{\psi}(s)$, the reasoning in the proof of part *1* applies.

State-space formulae for K^{ψ} in terms of the ψ-dependent matrices of the state-space realizations of N^{ψ} and M^{ψ} can be obtained directly from the expression (3.64) by repeated application of state space formulae for system inversion and series and parallel connection which are summarized at the beginning of this thesis in the section entitled **Notation, Definitions and Fundamental Results.** Continuous dependence on ψ of the resulting state-space realization for K^{ψ} can be verified from these formulae.

Suppose now that a ψ-dependent state-space realization of K^{ψ} is given. State-space formulae for V^{ψ}, as defined by application of the arguments in the proof of *1*, can be calculated directly from those for M^{ψ} and K^{ψ}, again by using the state space formulae for system inversion and series and parallel connection. Continuous dependence of this realization on ψ can again be verified from these formulae. From this state space description of V^{ψ} and the given realization of M^{ψ}, a state-space description of N^{ψ} follows from the last equality in the proof of part *1*. Continuous dependence of the state-space realization of N^{ψ} on ψ follows.

3. Since $W^{\psi}(s) \in \mathcal{RH}_{\infty}$ for each value of ψ we consider, it follows that $W^{\psi}(j\omega)$ is finite for all $\omega \in \mathbb{R} \cup \{\infty\}$. Consider any transfer function matrix $M(s) = D + C(sI - A)^{-1}B \in \mathcal{RH}_{\infty}$. It is easy to check that for any $\omega \in \mathbb{R} \cup \{\infty\}$, $M(j\omega)$ is locally a continuous function of the state-space matrices A, B, C, D. In particular, this is true of $W^{\psi}(s)$. Next recall that the state-space matrices in the given realization of $W^{\psi}(s)$ are assumed to be continuous functions of ψ. It follows that, for any given value of $\omega \in \mathbb{R} \cup \{\infty\}$, $W^{\psi}(j\omega)$ is a continuous function of ψ. Since the singular values are always a continuous function of the given matrix-valued data (see e.g. Appendix A of [97]), it follows that $\overline{\sigma}(W^{\psi}(j\omega))$ is a continuous function of ψ for all $\omega \in \mathbb{R} \cup \{\infty\}$.

It now follows from the definition $\|W^{\psi}\|_{\infty} = \sup_{\omega \in \mathbb{R} \cup \{\infty\}} \overline{\sigma}(W^{\psi}(j\omega))$ that $\|W^{\psi}\|_{\infty}$ is also a continuous function of ψ. □

Observe for future reference that:

- *For $\epsilon > 0$ it follows that $\tilde{M}^{\epsilon}$ in (3.58) satisfies the conditions in part 1 of Lemma 3.5.1.*

- *$\tilde{M}^{0}$ as defined in (3.60) also satisfies the conditions in part 1 of Lemma 3.5.1.*

3.5.2 A necessary Structure of all Nonstandard $\mathcal{H}_\infty$ Controllers.

In this subsection, we demonstrate that all nonstandard $\mathcal{H}_\infty$ control laws must *necessarily* have the structure of $\mathcal{K}_0^\gamma$. This proof follows in two steps. First we show that the nonstandard $\mathcal{H}_\infty$ control laws necessarily have the structure of a set $\overline{\mathcal{K}_0^\gamma}$ which is a superset of and is closely related to $\mathcal{K}_0^\gamma$. The second step of the proof is to show that is impossible for a nonstandard $\mathcal{H}_\infty$ controller to be in $\overline{\mathcal{K}_0^\gamma}$ and not in $\mathcal{K}_0^\gamma$.

Before stating the next result, recall the following: $M(s) \in \mathcal{RA}^+$ if $M(s)$ is real-rational and analytic in $\Re\{s\} > 0$; $M(s) \in \mathcal{RH}_\infty$ if $M(s)$ is real-rational and in $\mathcal{H}_\infty$; $M(s) \in \mathcal{RH}_\infty$ if and only if it is real-rational, proper and analytic in $\Re\{s\} \geq 0$; $M(s) \in \overline{\mathcal{BH}_\infty^\gamma}$ if $M(s) \in \mathcal{RH}_\infty$ and $\|M(s)\|_\infty \leq \gamma$.

Lemma 3.5.2 *Let $G(s)$ be a doubly nonstandard plant, realized as in (3.1), satisfying* **A.1** , **A.2** *and* **A.3** . *Let there exist at least one γ-admissible controller for $G(s)$. Let $\tilde{M}^0$ be defined as in Lemma 3.4.1 and define the set*

$$\overline{\mathcal{K}_0^\gamma} = \left\{K \mid K = LFT\{\tilde{M}^0, \begin{pmatrix} N & W_1 \\ W_2 & W_3 \end{pmatrix}\}, \quad N \in \overline{\mathcal{BH}_\infty^\gamma},\ W_i \in \mathcal{RA}^+ \ (i = 1,2,3)\right\}. \tag{3.66}$$

Then the set of all γ-admissible nonstandard controllers $\mathcal{K}^\gamma$ satisfies

$$\mathcal{K}^\gamma \subset \overline{\mathcal{K}_0^\gamma}.$$

Proof: Assuming $\mathcal{K}^\gamma$ is nonempty, take an arbitrary $K \in \mathcal{K}^\gamma$. Then by Lemma 3.2.2, $\exists\ \epsilon^*(K)$ such that $K \in \mathcal{K}_\epsilon^\gamma$ when $0 < \epsilon < \epsilon^*(K)$. Note that for each such ϵ, $\tilde{M}^\epsilon$ satisfies conditions *a)* and *b)* in part *1* of Lemma 3.5.1 and hence there is a *unique* $W^\epsilon \in \mathcal{RH}_\infty$ (see (3.56)) such that

$$K = LFT\{\tilde{M}^\epsilon, W^\epsilon\} = LFT\left\{\tilde{M}^\epsilon, \begin{pmatrix} N_{11}^\epsilon & W_1^\epsilon \\ W_2^\epsilon & W_3^\epsilon \end{pmatrix}\right\} \tag{3.67}$$

and since K is γ-admissible, it follows (recall (3.57)) that W^ϵ has the property:

$$\begin{pmatrix} N_{11}^\epsilon & \epsilon W_1^\epsilon \\ \epsilon W_2^\epsilon & \epsilon^2 W_3^\epsilon \end{pmatrix} = N^\epsilon \in \mathcal{BH}_\infty^\gamma. \tag{3.68}$$

Suppose one is given any state-space description of $K(s)$, along with the state space description of $\tilde{M}^\epsilon$ given in (3.58). Since the state-space realization of $\tilde{M}^\epsilon$ varies continuously with ϵ and K is fixed, it follows from part *2* of Lemma 3.5.1 that there are state-space realizations of N_{11}^ϵ and W_i^ϵ $(i = 1,2,3)$ which vary continuously with ϵ. Recall that $\tilde{M}^\epsilon \to \tilde{M}^0$ as $\epsilon \to 0$ and observe that $\tilde{M}^0$ also satisfies the conditions of part *1* of Lemma 3.5.1. Note therefore the existence of the limit

$$\lim_{\epsilon \to 0} \begin{pmatrix} N_{11}^\epsilon & W_1^\epsilon \\ W_2^\epsilon & W_3^\epsilon \end{pmatrix} = \begin{pmatrix} N & W_1 \\ W_2 & W_3 \end{pmatrix}.$$

It follows from (3.68) that $\forall\ \epsilon \in (0, \epsilon^*(K))$, $N_{11}^\epsilon \in \mathcal{BH}_\infty^\gamma$. Since the state-space matrices which describe N_{11}^ϵ vary continuously with ϵ, we can apply part *3* of Lemma 3.5.1 to conclude that $\|N\|_\infty \le \gamma$. Since $W_i^\epsilon \in \mathcal{RH}_\infty (i = 1, 2, 3)$ when $0 < \epsilon < \epsilon^*(K)$, it follows that $W_i \in \mathcal{RA}^+$. The stated result follows. □

In the following lemma, the above result is sharpened to exclude the possibility that a nonstandard γ-admissible controller could be on the "boundary" of the set $\overline{\mathcal{K}_0^\gamma}$. In other words, the aim is to extend the proof of the last lemma to show that $N \in \mathcal{BH}_\infty^\gamma$ and $W_i \in \mathcal{RH}_\infty (i = 1, 2, 3)$.

Lemma 3.5.3 *Let $G(s)$ be a doubly nonstandard plant, realized as in (3.1), satisfying* **A.1**, **A.2** *and* **A.3**. *Let $\bar{B}_1$ and $\bar{C}_1$ be chosen in accordance with Lemma 3.3.2. Let there exist at least one γ-admissible controller for $G(s)$.*
Let $\tilde{M}^0$ be defined as in Lemma 3.4.1 and define the set $\mathcal{K}_0^\gamma$ according to (3.63). Then the set of all γ-admissible nonstandard controllers $\mathcal{K}^\gamma$ satisfies

$$\mathcal{K}^\gamma \subset \mathcal{K}_0^\gamma.$$

Proof: Let K be any controller in $\mathcal{K}^\gamma$. Observe by Lemma 3.5.2 that

$$K = LFT\{\tilde{M}^0, \begin{pmatrix} N & W_1 \\ W_2 & W_3 \end{pmatrix}\} \tag{3.69}$$

for some $N \in \overline{\mathcal{BH}_\infty^\gamma}$ and $W_i \in \mathcal{RA}^+$ $(i = 1, 2, 3)$. Moreover, observe that we can always find a state-space realization of these matrices since they are real and rational.

For any N and W_i $(i = 1, 2, 3)$ as defined above, it will prove useful to consider the following perturbations of the control law $K(s)$, which depend on the two variables $\delta, \rho > 0$:

$$K^{\rho,\delta} = LFT\left\{\tilde{M}^0, \begin{pmatrix} N^{\rho,\delta} & W_1^{\rho,\delta} \\ W_2^{\rho,\delta} & W_3^{\rho,\delta} \end{pmatrix}\right\} = LFT\left\{\tilde{M}^0, \begin{pmatrix} (1+\delta)N(s-\rho) & W_1(s-\rho) \\ W_2(s-\rho) & W_3(s-\rho) \end{pmatrix}\right\}, \tag{3.70}$$

and note that $\lim_{\delta,\rho\to 0} K^{\rho,\delta} = K$. Note that the perturbed transfer function matrices $N^{\rho,\delta}$ and $W_i^{\rho,\delta}$ $(i = 1, 2, 3)$ have been chosen in a manner such that they will *violate* the constraints $N^{\rho,\delta} \in \mathcal{BH}_\infty^\gamma$ and $W_i^{\rho,\delta} \in \mathcal{RH}_\infty$ $(i = 1, 2, 3)$ for sufficiently large ρ and/or δ.

Let $\epsilon \in (0, \epsilon^*(K))$ where $\epsilon^*(K)$ is defined by application of Lemma 3.2.2 to K. Since $\tilde{M}^\epsilon$ satisfies assumptions *a)* and *b)* in part *1* of Lemma 3.5.1, $W^{\rho,\delta,\epsilon}$,$N_{11}^{\rho,\delta,\epsilon}$ and $W_i^{\rho,\delta,\epsilon}$ $(i = 1, 2, 3)$ are uniquely defined via the following equalities:

$$K^{\rho,\delta} = LFT\left\{\tilde{M}^\epsilon, W^{\rho,\delta,\epsilon}\right\}, \tag{3.71}$$

$$W^{\rho,\delta,\epsilon} = \begin{pmatrix} N_{11}^{\rho,\delta,\epsilon} & W_1^{\rho,\delta,\epsilon} \\ W_2^{\rho,\delta,\epsilon} & W_3^{\rho,\delta,\epsilon} \end{pmatrix}, \tag{3.72}$$

with block partitioning compatible with that of the coefficient matrix M^ϵ of the linear fractional map above.

Given a state space realization of N and W_i $(i = 1, 2, 3)$, item *2* of Lemma 3.5.1 confirms the existence of a state space realization of $K^{\rho,\delta}$ defined via (3.70), whose constituent matrices are continuous functions of ρ and δ.

Since there exist state-space realizations of both $K^{\rho,\delta}$ and $\tilde{M}^\epsilon$ which are continuous functions of ρ, δ and ϵ, it follows by part *2* of Lemma 3.5.1 that there exists a state-space realization of $W^{\rho,\delta,\epsilon}$ and thus also of $N_{11}^{\rho,\delta,\epsilon}$ and $W_i^{\rho,\delta,\epsilon}$ $(i = 1,2,3)$, which vary continuously with ϵ, δ and ρ.

Observe by continuity that for any fixed ϵ,

$$\lim_{\delta,\rho\to 0} \begin{pmatrix} N_{11}^{\rho,\delta,\epsilon} & W_1^{\rho,\delta,\epsilon} \\ W_2^{\rho,\delta,\epsilon} & W_3^{\rho,\delta,\epsilon} \end{pmatrix} = \begin{pmatrix} N_{11}^\epsilon & W_1^\epsilon \\ W_2^\epsilon & W_3^\epsilon \end{pmatrix} = W^\epsilon$$

where W^ϵ is the unique transfer function matrix which is defined by inversion of the linear fractional map in (3.67).

Recall that $\epsilon \in (0, \epsilon^*(K))$. From the alternative expression of all ϵ-controllers given in (3.55), (3.56) and from (3.57) in particular it follows that

$$\begin{pmatrix} N_{11}^\epsilon & \epsilon W_1^\epsilon \\ \epsilon W_2^\epsilon & \epsilon^2 W_3^\epsilon \end{pmatrix} = N^\epsilon \in \mathcal{BH}_\infty^\gamma. \tag{3.73}$$

Since it was established above that $W^{\rho,\delta,\epsilon}$ as defined by (3.71) has a state-space realization which is a continuous function of each of the parameters ρ, δ and ϵ, the following transfer function matrix also has a state-space realization continuous in each of these parameters:

$$N^{\rho,\delta,\epsilon} = \begin{pmatrix} I & 0 \\ 0 & \epsilon I \end{pmatrix} \begin{pmatrix} N_{11}^{\rho,\delta,\epsilon} & W_1^{\rho,\delta,\epsilon} \\ W_2^{\rho,\delta,\epsilon} & W_3^{\rho,\delta,\epsilon} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \epsilon I \end{pmatrix} = \begin{pmatrix} N_{11}^{\rho,\delta,\epsilon} & \epsilon W_1^{\rho,\delta,\epsilon} \\ \epsilon W_2^{\rho,\delta,\epsilon} & \epsilon^2 W_3^{\rho,\delta,\epsilon} \end{pmatrix}.$$

By applying item *3* of Lemma 3.5.1 to this realization of $N^{\rho,\delta,\epsilon}$ and by recognizing the property (3.73), it follows that $\exists\ \rho^*(K,\epsilon) > 0$ and $\delta^*(K,\epsilon) > 0$ such that provided both $0 < \rho < \rho^*(K,\epsilon)$ and $0 < \delta < \delta^*(K,\epsilon)$,

$$N^{\rho,\delta,\epsilon} \in \mathcal{BH}_\infty^\gamma.$$

With the above restrictions on ρ and δ, it then follows from (3.71) that $K^{\rho,\delta} \in \mathcal{K}_\epsilon^\gamma$. This follows since we have shown that $K^{\rho,\delta}$ can be expressed according to the alternative form for ϵ-controllers as described by the identities (3.55), (3.56) and (3.57).

Since $\mathcal{K}_\epsilon^\gamma \subset \mathcal{K}^\gamma$ for any $\epsilon > 0$, we conclude that $K^{\rho,\delta} \in \mathcal{K}^\gamma$, provided both $0 < \rho < \rho^*(K,\epsilon)$ and $0 < \delta < \delta^*(K,\epsilon)$. From Lemma 3.5.2, it follows that $K^{\rho,\delta} \in \overline{\mathcal{K}_0^\gamma}$.

We now establish that $K \in \mathcal{K}_0^\gamma$ by contradiction:
recall again that $K = LFT\{\tilde{M}^0, \begin{pmatrix} N & W_1 \\ W_2 & W_3 \end{pmatrix}\}$ for some $N \in \overline{\mathcal{BH}_\infty^\gamma}$ and $W_i \in \mathcal{RA}^+$ $(i = 1,2,3)$. Suppose that $K \in \overline{\mathcal{K}_0^\gamma} \backslash \mathcal{K}_0^\gamma$; then either $\|N\|_\infty = \gamma$ or W_i has a $j\omega$-axis pole for some i (or both). If this is the case, then it can then be easily seen from the definition of $K^{\rho,\delta}$ that at least one of the following conditions hold $\forall\ \delta, \rho > 0$:

1. $\|N^{\rho,\delta}\|_\infty > \gamma$.

2. At least one of $W_i^{\rho,\delta} \notin \mathcal{RA}^+$ $(i = 1, 2, 3)$.

This allows one to conclude that $K^{\rho,\delta} \notin \overline{\mathcal{K}_0^\gamma}$, which contradicts the previously established conclusion that $K^{\rho,\delta} \in \overline{\mathcal{K}_0^\gamma}$. Thus we conclude that $K \in \mathcal{K}_0^\gamma$. □

3.5.3 The Controller Structure is also sufficient.

The following lemma establishes the reverse inclusion to that of Lemma 3.5.3, and thereby proves the theorem stated earlier which says that $\mathcal{K}_0^\gamma = \mathcal{K}^\gamma$.

Lemma 3.5.4 *Let $G(s)$ be a doubly nonstandard plant, realized as in (3.1), satisfying* **A.1** *,* **A.2** *and* **A.3** *. Let there exist at least one γ-admissible controller for $G(s)$. Let $\tilde{M}^0$ be defined by Lemma 3.4.1 and define the set $\mathcal{K}_0^\gamma$ according to (3.63), then*

$$\mathcal{K}_0^\gamma \subset \mathcal{K}^\gamma$$

where $\mathcal{K}^\gamma$ denotes the set of all γ-admissible controllers for $G(s)$.

Proof: Take any $K_0 \in \mathcal{K}_0^\gamma$. By definition, there exists an $N \in \mathcal{BH}_\infty^\gamma$ and $W_i \in \mathcal{RH}_\infty$ $(i = 1, 2, 3)$ such that

$$K_0 = LFT\left\{\tilde{M}^0, \begin{pmatrix} N & W_1 \\ W_2 & W_3 \end{pmatrix}\right\}. \tag{3.74}$$

Note that $\tilde{M}^0$ satisfies the assumptions in part *1* of Lemma 3.5.1 and thus N and W_i $(i = 1, 2, 3)$ are unique.

Note that $\tilde{M}^\epsilon$ also satisfies the assumptions in part *1* of Lemma 3.5.1 for any $\epsilon > 0$. A consequence of this is that for any $\epsilon > 0$, the following equality uniquely defines a transfer function matrix $W^\epsilon(s)$ satisfying:

$$K_0 = LFT\{\tilde{M}^\epsilon, W^\epsilon\} = LFT\left\{\tilde{M}^\epsilon, \begin{pmatrix} N_{11}^\epsilon & W_1^\epsilon \\ W_2^\epsilon & W_3^\epsilon \end{pmatrix}\right\}. \tag{3.75}$$

Since K_0 is fixed and the state-space realization of $\tilde{M}^\epsilon$ is a continuous function of ϵ (something guaranteed by Lemma 3.4.1), one can apply part *2* of Lemma 3.5.1 to (3.75) to reveal that W^ϵ (and each of its partitions) also has a state-space realization which is a continuous function of ϵ.

Observe also that since $\tilde{M}^\epsilon \to \tilde{M}^0$, it follows from (3.75) together with (3.74) that

$$\lim_{\epsilon \to 0} \begin{pmatrix} N_{11}^\epsilon & W_1^\epsilon \\ W_2^\epsilon & W_3^\epsilon \end{pmatrix} = \begin{pmatrix} N & W_1 \\ W_2 & W_3 \end{pmatrix}.$$

Note that since each of the component matrices of this limit are stable by hypothesis, it follows that there exists an $\bar{\epsilon}(K_0) > 0$ such that when $\bar{\epsilon}(K_0) > \epsilon > 0$, W^ϵ and each of its partitions are in $\mathcal{RH}_\infty$.

With the definition

$$N^\epsilon = \begin{pmatrix} I & 0 \\ 0 & \epsilon \end{pmatrix} \begin{pmatrix} N_{11}^\epsilon & W_1^\epsilon \\ W_2^\epsilon & W_3^\epsilon \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \epsilon \end{pmatrix},$$

note that

$$\lim_{\epsilon \to 0} N^\epsilon = \begin{pmatrix} N & 0 \\ 0 & 0 \end{pmatrix},$$

where N is defined according to the identity (3.74). Since N is, from the definition of $\mathcal{K}_0^\gamma$, in $\mathcal{BH}_\infty^\gamma$, we can apply part *3* of Lemma 3.5.1 to N^ϵ, to conclude that $\exists\ \hat{\epsilon}(K_0)$ s.t. when $0 < \epsilon < \hat{\epsilon}(K_0)$,

$$\begin{pmatrix} N_{11}^\epsilon & \epsilon W_1^\epsilon \\ \epsilon W_2^\epsilon & \epsilon^2 W_3^\epsilon \end{pmatrix} = N^\epsilon \in \mathcal{BH}_\infty^\gamma. \tag{3.76}$$

Thus with $\tilde{\epsilon}(K_0) = \min(\hat{\epsilon}(K_0), \bar{\epsilon}(K_0))$, $\epsilon \in (0, \tilde{\epsilon}(K_0))$ guarantees that both $W^\epsilon \in \mathcal{RH}_\infty$ and that the constraint (3.76) holds. These facts, together with the identity (3.75), are enough to ensure that $K_0 \in \mathcal{K}_\epsilon^\gamma$. This follows from the discussion of the alternative form for $\mathcal{H}_\infty$ controllers given in subsection 3.4.1. Recall from Lemma 3.2.1 that for any $\epsilon > 0$, $\mathcal{K}_\epsilon^\gamma \subset \mathcal{K}^\gamma$. The stated result that $\mathcal{K}_0^\gamma \subset \mathcal{K}^\gamma$ follows directly from the last two observations. □

Lemmas 3.5.3 and 3.5.4 together establish Theorem 3.5.1 which is the main result of this section; $\mathcal{K}_0^\gamma = \mathcal{K}^\gamma$. The explicit construction of this set given in the Theorem statement constitutes a full state-space characterization of the set of all γ-admissible controllers for the doubly nonstandard (case 2) plant $G(s)$. We shall not be concerned with the proofs for singly nonstandard plants (cases 1 and 3) since they follow by completely analogous means, and are somewhat simpler than the proof for the doubly nonstandard case. The controller existence and parametrization results for cases 1 and 3 are summarized in the next section.

3.6 Summary of the Main Results.

The $\mathcal{H}_\infty$ controller existence and parametrization results for case 1,2 and 3 nonstandard plants are summarized in this section. These results, together with the standard (case 0) results (see section 2.3), constitute a full solution of the state space $\mathcal{H}_\infty$ problem without signal dimension restrictions which was introduced in Chapter 2. We now recount the problem statement and assumptions for convenient reference.

Given a generalized plant $G(s)$, realized as follows:

$$G(s) = \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array} \right), \tag{3.77}$$

which satisfies the following assumptions:

A.1 (A, B_2) is stabilizable and (C_2, A) is detectable.

A.2 Both D_{12} and D_{21} are full rank.

A.3 Neither $G_{12}(s)$ nor $G_{21}(s)$, as described by the state space realization in (3.77), have imaginary axis invariant zeros.

Find all FDLTI control laws $K(s)$ which ensure that the closed loop mapping $T_{zw}^K(s)$ is internally stable and $\|T_{zw}^K\|_\infty < \gamma$.

Riccati Equations for Controller Existence and Parametrization.

The algebraic Riccati equations required in the expression of the main results of the present Chapter are summarized next. Two of these equations have already been introduced in this chapter in the doubly nonstandard derivation. The other two equations are the AREs which appear in the standard results summarized in section 2.3. These equations are required in the expression of the singly nonstandard results.

- When $\boldsymbol{D_{12}}$ **is standard** (i.e. $n_z \geq n_u$), consider the equation

$$\begin{aligned} 0 = &X_0(A - B_2D_{12}{}^\dagger C_1) + (A - B_2D_{12}{}^\dagger C_1)^T X_0 \\ &+ X_0(\gamma^{-2}B_1B_1^T - B_2D_{12}{}^\dagger(D_{12}{}^\dagger)^T B_2^T)X_0 + C_1^T(D_{12}{}^\perp)^T D_{12}{}^\perp C_1, \end{aligned} \tag{3.78}$$

where $D_{12}{}^\dagger$ and $D_{12}{}^\perp$ satisfy the following relation

$$\begin{pmatrix} D_{12}{}^\dagger \\ D_{12}{}^\perp \end{pmatrix} \begin{pmatrix} D_{12} & (D_{12}{}^\perp)^T \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{3.79}$$

- When $\boldsymbol{D_{12}}$ **is nonstandard** (i.e. $n_z < n_u$), consider the equation

$$X_0A_{ZX} + A_{ZX}{}^T X_0 + X_0(\gamma^{-2}B_1B_1^T - B_2D_{12}{}^\dagger(D_{12}{}^\dagger)^T B_2^T)X_0 = 0, \tag{3.80}$$

with $A_{ZX} = A - B_2D_{12}{}^\dagger C_1 - B_2D_{12}{}^\perp \bar{C}_1$, where $D_{12}{}^\dagger$ and $D_{12}{}^\perp$ satisfy the following relation

$$\begin{pmatrix} D_{12} \\ (D_{12}{}^\perp)^T \end{pmatrix} \begin{pmatrix} D_{12}{}^\dagger & D_{12}{}^\perp \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \tag{3.81}$$

and where $\bar{C}_1$ is chosen according to Lemma 3.3.2 such that it stabilizes the controllable subspace of the pair $(A - B_2D_{12}{}^\dagger C_1, -B_2D_{12}{}^\perp)$.

- When $\boldsymbol{D_{21}}$ **is standard** (i.e. $n_y \leq n_w$), consider the equation

$$\begin{aligned} 0 = &Y_0(A - B_1D_{21}{}^\dagger C_2)^T + (A - B_1D_{21}{}^\dagger C_2)Y_0 \\ &+ Y_0(\gamma^{-2}C_1^TC_1 - C_2^T(D_{21}{}^\dagger)^T D_{21}{}^\dagger C_2)Y_0 + B_1D_{21}{}^\perp(D_{21}{}^\perp)^T B_1^T, \end{aligned} \tag{3.82}$$

where $D_{21}{}^\dagger$ and $D_{21}{}^\perp$ satisfy the following relation

$$\begin{pmatrix} D_{21} \\ (D_{21}{}^\perp)^T \end{pmatrix} \begin{pmatrix} D_{21}{}^\dagger & D_{21}{}^\perp \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{3.83}$$

- When $\boldsymbol{D_{21}}$ **is nonstandard** (i.e. $n_y > n_w$), consider the equation

$$Y_0A_{ZY}{}^T + A_{ZY}Y_0 + Y_0(\gamma^{-2}C_1^TC_1 - C_2^T(D_{21}{}^\dagger)^T D_{21}{}^\dagger C_2)Y_0 = 0, \tag{3.84}$$

with $A_{ZY} = A - B_1 D_{21}{}^{\dagger} C_2 - \bar{B}_1 D_{21}{}^{\perp} C_2$, where $D_{21}{}^{\dagger}$ and $D_{21}{}^{\perp}$ satisfy the following relation

$$\begin{pmatrix} D_{21}{}^{\dagger} \\ D_{21}{}^{\perp} \end{pmatrix} \begin{pmatrix} D_{21} & (D_{21}{}^{\perp})^T \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \tag{3.85}$$

and where $\bar{B}_1$ is chosen according to Lemma 3.3.2 such that it stabilizes the observable subspace of the pair $(-D_{21}{}^{\perp} C_2, A - B_1 D_{21}{}^{\dagger} C_2)$.

Nonstandard $\mathcal{H}_\infty$ Controller Existence and Parametrization Results.

Theorem 3.6.1 (Case 1) *Given a singly nonstandard generalized plant (3.77), satisfying assumptions* **A.1, A.2** *and* **A.3** *for which D_{12} is nonstandard and D_{21} is standard, let $D_{12}{}^{\dagger}$ and $D_{12}{}^{\perp}$ satisfy (3.81) and let $D_{21}{}^{\dagger}$ and $D_{21}{}^{\perp}$ satisfy (3.83). Let $\bar{C}_1$ be chosen such that it stabilizes the controllable subspace of the pair $(A - B_2 D_{12}{}^{\dagger} C_1, -B_2 D_{12}{}^{\perp})$.*

Then a γ-admissible controller for $G(s)$ exists if and only if the AREs (3.80) and (3.82) have stabilizing solutions $X_0 \geq 0$ and $Y_0 \geq 0$ which satisfy $\rho(X_0 Y_0) < \gamma^2$.

Moreover, under such conditions, every γ-admissible controller $K(s)$ can be expressed as:

$$K(s) = LFT\left\{ \left(\begin{array}{c|c:cc} \hat{A} + B_2 F_\infty + H_\infty \hat{C}_2 & -H_\infty & Z_0 \hat{B}_2 D_{12}{}^{\dagger} & B_2 D_{12}{}^{\perp} \\ \hline F_\infty & 0 & D_{12}{}^{\dagger} & D_{12}{}^{\perp} \\ \hdashline -V_{21} \hat{C}_2 & V_{21} & 0 & 0 \end{array} \right), \begin{pmatrix} N \\ W \end{pmatrix} \right\} \tag{3.86}$$

with free parameters

$$N(s) \in \mathcal{BH}_\infty^{\prime}, \quad W(s) \in \mathcal{RH}_\infty.$$

Here V_{21} is any square matrix (not necessarily symmetric) which satisfies

$$V_{21}^T V_{21} = (D_{21} D_{21}^T)^{-1}$$

and the remaining matrices are defined as

$$\begin{aligned} F_\infty &= -D_{12}{}^{\dagger} C_1 - D_{12}{}^{\perp} \bar{C}_1 - D_{12}{}^{\dagger} (D_{12}{}^{\dagger})^T B_2{}^T X_0, & \hat{A} &= A + \gamma^{-2} B_1 B_1^T X_0, \\ H_\infty &= Z_0 \left(-B_1 D_{21}{}^{\dagger} - Y_0 C_2^T (D_{21}{}^{\dagger})^T D_{21}{}^{\dagger} \right), & \hat{C}_2 &= C_2 + \gamma^{-2} D_{21} B_1^T X_0, \\ Z_0 &= (I - \gamma^{-2} Y_0 X_0)^{-1}, & \hat{B}_2 &= B_2 + \gamma^{-2} Y_0 C_1^T D_{12}. \end{aligned}$$

Proof: This result can be proven in a manner completely analogous to that for the doubly nonstandard plant. The details of the proof are therefore omitted. □

Theorem 3.6.2 (Case 2) *Given a doubly nonstandard generalized plant (3.77), satisfying assumptions* **A.1, A.2** *and* **A.3** *and for which both D_{12} and D_{21} are nonstandard, let $D_{21}{}^{\dagger}$ and $D_{21}{}^{\perp}$ satisfy (3.81) and let $D_{12}{}^{\dagger}$ and $D_{12}{}^{\perp}$ satisfy (3.85). Let $\bar{C}_1$ be chosen* *such that it stabilizes the controllable subspace of the pair* $(A - B_2 D_{12}{}^{\dagger} C_1, -B_2 D_{12}{}^{\perp})$ *and let* $\bar{B}_1$ *be chosen such that it stabilizes the observable subspace of the pair* $(-D_{21}{}^{\perp} C_2, A - B_1 D_{21}{}^{\dagger} C_2)$.

Then a γ*-admissible controller for* $G(s)$ *exists if and only if the AREs (3.80) and (3.84) have stabilizing solutions* $X_0 \geq 0$ *and* $Y_0 \geq 0$ *which satisfy* $\rho(X_0 Y_0) < \gamma^2$.

Moreover, under such conditions, every γ*-admissible controller* $K(s)$ *can be expressed as:*

$$K(s) = \tag{3.87}$$
$$LFT\left\{\left(\begin{array}{c|c:cc} \hat{A} + B_2 F_\infty + H_\infty \hat{C}_2 & -H_\infty & Z_0 \hat{B}_2 D_{12}{}^{\dagger} & B_2 D_{12}{}^{\perp} \\ \hline F_\infty & 0 & D_{12}{}^{\dagger} & D_{12}{}^{\perp} \\ \hdashline -D_{21}{}^{\dagger} \hat{C}_2 & D_{21}{}^{\dagger} & 0 & 0 \\ -D_{21}{}^{\perp} C_2 & D_{21}{}^{\perp} & 0 & 0 \end{array}\right), \begin{pmatrix} N & W_1 \\ W_2 & W_3 \end{pmatrix}\right\}$$

with free parameters

$$N(s) \in \mathcal{BH}_\infty^{\Omega}\ ,\quad W_i(s) \in \mathcal{RH}_\infty\ (i = 1, 2, 3),$$

and the definitions

$$\begin{array}{rcl} F_\infty & = & -D_{12}{}^{\dagger} C_1 - D_{12}{}^{\perp} \bar{C}_1 - D_{12}{}^{\dagger} (D_{12}{}^{\dagger})^T B_2{}^T X_0, \\ H_\infty & = & Z_0(-B_1 D_{21}{}^{\dagger} - \bar{B}_1 D_{21}{}^{\perp} - Y_0 C_2^T (D_{21}{}^{\dagger})^T D_{21}{}^{\dagger}), \\ Z_0 & = & (I - \gamma^{-2} Y_0 X_0)^{-1}, \end{array} \qquad \begin{array}{rcl} \hat{A} & = & A + \gamma^{-2} B_1 B_1^T X_0, \\ \hat{C}_2 & = & C_2 + \gamma^{-2} D_{21} B_1^T X_0, \\ \hat{B}_2 & = & B_2 + \gamma^{-2} Y_0 C_1^T D_{12}. \end{array}$$

Proof: The proof of this result is presented in the body of the present chapter. □

Theorem 3.6.3 (Case 3) *Given a singly nonstandard generalized plant (3.77), satisfying assumptions* **A.1**, **A.2** *and* **A.3** *and for which* D_{12} *is standard and* D_{21} *is nonstandard, let* $D_{12}{}^{\dagger}$ *and* $D_{12}{}^{\perp}$ *satisfy (3.79) and* $D_{21}{}^{\dagger}$ *and* $D_{21}{}^{\perp}$ *satisfy (3.85). Let* $\bar{B}_1$ *be chosen such that it stabilizes the observable subspace of the pair* $(-D_{21}{}^{\perp} C_2, A - B_1 D_{21}{}^{\dagger} C_2)$.

Then a γ*-admissible controller for* $G(s)$ *exists if and only if (3.78) and (3.84) have stabilizing solutions* $X_0 \geq 0$ *and* $Y_0 \geq 0$ *which satisfy* $\rho(X_0 Y_0) < \gamma^2$.

Moreover, under such conditions, every γ*-admissible controller* $K(s)$ *can be expressed as:*

$$K(s) = LFT\left\{\left(\begin{array}{c|c:c} \tilde{A} + \hat{B}_2 F_\infty + H_\infty C_2 & -H_\infty & \hat{B}_2 V_{12} \\ \hline F_\infty & 0 & V_{12} \\ \hdashline -D_{21}{}^{\dagger} \hat{C}_2 Z_0 & D_{21}{}^{\dagger} & 0 \\ -D_{21}{}^{\perp} C_2 & D_{21}{}^{\perp} & 0 \end{array}\right), \begin{pmatrix} N & W \end{pmatrix}\right\} \tag{3.88}$$

with free parameters

$$N(s) \in \mathcal{BH}_\infty^{\Omega}\ ,\quad W(s) \in \mathcal{RH}_\infty.$$

Here V_{12} is any square matrix (not necessarily symmetric) which satisfies

$$V_{12}V_{12}^T = (D_{12}^T D_{12})^{-1}$$

and the remaining matrices are defined as

$$\begin{aligned} F_\infty &= (-D_{12}{}^\dagger C_1 - D_{12}{}^\dagger (D_{12}{}^\dagger)^T B_2^T X_0) Z_0, & \tilde{A} &= A + \gamma^{-2} Y_0 C_1^T C_1, \\ H_\infty &= -B_1 D_{21}{}^\dagger - \bar{B}_1 D_{21}{}^\perp - Y_0 C_2^T (D_{21}{}^\dagger)^T D_{21}{}^\dagger, & \hat{C}_2 &= C_2 + \gamma^{-2} D_{21} B_1^T X_0, \\ Z_0 &= (I - \gamma^{-2} Y_0 X_0)^{-1}, & \hat{B}_2 &= B_2 + \gamma^{-2} Y_0 C_1^T D_{12}. \end{aligned}$$

Proof: This result follows by application of the case 1 result to the transpose of the case 3 plant. Transposition of the resulting existence conditions and controller formulae give the above result. □

3.7 Conclusions about the Plant Augmentation Approach.

This chapter has addressed the nonstandard cases of the state-space $\mathcal{H}_\infty$ controller synthesis problem without signal dimension restrictions which was introduced in Chapter 2. A special case of this problem is the so-called standard problem (case 0) in which an additional assumption is made concerning the comparative dimensions of some of the input and output signal spaces of the generalized plant. Results for the nonstandard plants (cases 1–3) have been presented in this chapter which, together with the standard result (case 0), constitute a full solution of the state space $\mathcal{H}_\infty$ problem posed in Chapter 2.

The approach developed to solve the nonstandard problems relies on a parametrized augmentation of the nonstandard plant followed by application of standard $\mathcal{H}_\infty$ results and limiting arguments. A detailed proof has only been provided for case 2 (doubly nonstandard) plants since completely analogous and somewhat simplified reasoning is required to prove the case 1 and case 3 (singly nonstandard) results.

Aside from the preliminary calculation of the augmentation matrices $\bar{C}_1$ and/or $\bar{B}_1$ (which involve standard manipulations only), the computational burden associated with the nonstandard results is the same as that for the standard case. As in the standard case, the resulting existence conditions depend on the solution of two algebraic Riccati equations and the satisfaction of a bound on the spectral radius of their product. Analogous to the standard case, the set of all γ-admissible controllers can be expressed in terms of a linear fractional transformation with a coefficient matrix which can be constructed using the Riccati equation solutions. It is well known that standard $\mathcal{H}_\infty$ controllers are parametrized by a free bounded real transfer function matrix. In the nonstandard case, additional degrees of freedom in the form of stable transfer function matrices.

Further discussion of the nonstandard results is postponed until section 4.6 of Chapter 4. In Chapter 4, an alternative approach is presented to solving the same nonstandard $\mathcal{H}_\infty$ problems. Section 4.6 also contains a comparison of the results obtained via the two different approaches.

Chapter 4

Nonstandard $\mathcal{H}_\infty$ Synthesis via a Lossless Decomposition and the Youla Parametrization.

Summary.

Like Chapter 3, the present chapter addresses the nonstandard state-space $\mathcal{H}_\infty$ control problem without signal dimension restrictions, which was introduced in Chapter 2. Both existence conditions and controller parametrizations are presented for each type of nonstandard generalized plant. As in Chapter 3, controller existence results are expressed in terms of a pair of algebraic Riccati equations, together with a constraint on the spectral radius of the product of the solutions to these equations. The derivation of the existence result employs the quadratic matrix inequalities and associated rank conditions which were derived for a more general (singular) $\mathcal{H}_\infty$ problem in [98]. The controller derivation presented in this chapter draws upon the well-known state-space results for the parametrization of all stabilizing controllers as well as on the parametrization of all standard $\mathcal{H}_\infty$ controllers. Controller parametrizations are derived by reducing the doubly nonstandard problem to a related standard $\mathcal{H}_\infty$ problem. The first step in this process is to construct the so-called *temporary* generalized plant via a lossless transformation of the original generalized plant. The temporary plant is equivalent to the original plant in the sense that the family of $\mathcal{H}_\infty$ controllers is identical for both. In this sense, the approach bears close similarities with the well known derivation of standard $\mathcal{H}_\infty$ controllers (see e.g. [32]). In contrast with the standard case however, the temporary plant is not directly amenable to application of standard output estimation $\mathcal{H}_\infty$ results.

Instead, a state-space construction of the Youla parametrization is next applied to describe all stabilizing controllers for the temporary plant. A new and equivalent $\mathcal{H}_\infty$ problem is then posed; choose the Youla parameter such that the closed loop for the temporary plant is bounded real. Introducing the temporary plant has the important consequence that it enables all internally stable closed loop systems to be described as a linear fractional map with a coefficient matrix which has the same order as that of

the plant, when this coefficient matrix would otherwise normally have twice the plant order. It turns out that the new and equivalent $\mathcal{H}_\infty$ problem *can* be easily reduced to a standard *output estimation* $\mathcal{H}_\infty$ synthesis problem, to which the well-known standard state-space $\mathcal{H}_\infty$ results of [32] can be applied. The result is a state-space description of the required subset of Youla parameters. A formula for all $\mathcal{H}_\infty$ control laws for the temporary generalized plant is then derived by substituting the description of this subset of Youla parameters into the parametrization of all controllers which stabilize the generalized plant.

The final controller form differs slightly from that which was derived using the parametrized plant augmentation approach described in Chapter 3. However, the parametrizations are equivalent in that they both describe all $\mathcal{H}_\infty$ control laws. This equivalence is established in the final section of this chapter. The chapter concludes with an interpretation of the nonstandard state space $\mathcal{H}_\infty$ results. The proof techniques used in each approach are also compared.

4.1 Controller Existence Conditions.

Recall the class of state space realizations of generalized plants which were introduced in Chapter 2:

$$G(s) = \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right). \tag{4.1}$$

Recall also that we make the following assumptions on the realization of $G(s)$:

A.1 (A, B_2) is stabilizable and (C_2, A) is detectable.

A.2 Both D_{12} and D_{21} are full rank.

A.3 Neither $G_{12}(s)$ nor $G_{21}(s)$, as described by the state space realization in (4.1), have imaginary axis invariant zeros.

Existence conditions for a more general class of so-called *singular* problems are considered in [98], where no assumptions are made on either the rank or shape of the feedthrough matrices D_{12} or D_{21}. The conditions obtained there are expressed in terms of *quadratic matrix inequalities* and are summarized in Lemma 4.1.1 in the next subsection. The objective in this section is to specialize the result of [98] to the class of plants where assumption **A.2** holds in addition to assumptions **A.1** and **A.3**. The main conclusion is that the existence conditions of [98] can be expressed equivalently in terms of the existence of nonnegative definite stabilizing solutions of a pair of *algebraic Riccati equations* which are easily solvable using standard software. This constitutes an important insight since software, especially commercial software, for solving quadratic matrix inequalities is not in widespread use.

Only doubly nonstandard (case 2) plants, as defined in subsection 2.6 of Chapter 2, are treated. Proofs for the singly nonstandard case 1 and case 3 generalized plants follow completely analogous reasoning. For simplicity throughout this chapter, we set $\gamma = 1$, without loss of generality. Having established results for the $\gamma = 1$ case, they can easily be applied with $\gamma \neq 1$ by simple scaling of the state-space data. For example, one can apply results for $\gamma = 1$ to a generalized plant which has B_1 replaced by $\gamma^{-1}B_1$ and D_{21} replaced by $\gamma^{-1}D_{21}$.

4.1.1 Singular $\mathcal{H}_\infty$ Existence Conditions via Quadratic Matrix Inequalities.

For reference, we restate the main result of [98] which presents necessary and sufficient conditions for the existence of $\mathcal{H}_\infty$ controllers in terms of a pair of quadratic matrix inequalities. We now recall Theorem 2.1 of that paper, incorporating minor changes in notation and emphasis.

Adopting the notation introduced in (4.1) for the generalized plant, we define four matrix valued functions of the $n \times n$ square matrices X and Y, where n is the dimension of the state space of the generalized plant:

$$
\begin{aligned}
F(X) &= \begin{pmatrix} A^TX + XA + C_1^TC_1 + XB_1B_1^TX & XB_2 + C_1^TD_{12} \\ B_2^TX + D_{12}^TC_1 & D_{12}^TD_{12} \end{pmatrix}, && (4.2)\\
H(Y) &= \begin{pmatrix} AY + YA^T + B_1B_1^T + YC_1^TC_1Y & YC_2^T + B_1D_{21}^T \\ C_2Y + D_{21}B_1^T & D_{21}D_{21}^T \end{pmatrix}, && (4.3)\\
L(X,s) &= \begin{pmatrix} sI - A - B_1B_1^TX & -B_2 \end{pmatrix}, && (4.4)\\
M(Y,s) &= \begin{pmatrix} sI - A - YC_1^TC_1 \\ -C_2 \end{pmatrix}. && (4.5)
\end{aligned}
$$

Lemma 4.1.1 *Given a generalized plant $G(s)$ with state space realization (4.1) satisfying assumptions* **A.1** *and* **A.3**, *an $\mathcal{H}_\infty$ controller exists if and only if there exist solutions $X \geq 0$ and $Y \geq 0$ to the quadratic matrix inequalities $F(X) \geq 0$ and $H(Y) \geq 0$ which satisfy $\rho(XY) < 1$ and for which*

1. $\operatorname{rank}\{F(X)\} = \operatorname{normrank}\{G_{12}(s)\}$.
2. $\operatorname{rank}\{H(Y)\} = \operatorname{normrank}\{G_{21}(s)\}$.
3. *When $\Re\{s\} \geq 0$,*

$$\operatorname{rank}\begin{pmatrix} L(X,s) \\ F(X) \end{pmatrix} = n + \operatorname{normrank}\{G_{12}(s)\}. \qquad (4.6)$$

4. *When $\Re\{s\} \geq 0$,*

$$\operatorname{rank}\begin{pmatrix} M(Y,s) & H(Y) \end{pmatrix} = n + \operatorname{normrank}\{G_{21}(s)\}. \qquad (4.7)$$

The objective of the present section is to establish necessary and sufficient conditions for the existence of nonstandard controllers by using the above quadratic matrix inequalities and rank conditions. We shall thus examine these conditions for the special case where assumption **A.2** is *maintained*, i.e. the feedthrough matrices D_{12} and D_{21} are full rank.

4.1.2 Statement of the Nonstandard Controller Existence Conditions.

Recall that for doubly nonstandard (case 2) plants, $n_u > n_z$ and $n_w < n_y$. Recall also from section 3.1 of Chapter 3 how matrices $D_{12}{}^{\dagger}$ and $D_{12}{}^{\perp}$ can be constructed which satisfy the following relation:

$$\begin{pmatrix} D_{12} \\ (D_{12}{}^{\perp})^T \end{pmatrix} \begin{pmatrix} D_{12}{}^{\dagger} & D_{12}{}^{\perp} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{4.8}$$

Likewise, recall how matrices $D_{21}{}^{\dagger}$ and $D_{21}{}^{\perp}$ can be constructed which satisfy the following relation:

$$\begin{pmatrix} D_{21}{}^{\dagger} \\ D_{21}{}^{\perp} \end{pmatrix} \begin{pmatrix} D_{21} & (D_{21}{}^{\perp})^T \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{4.9}$$

We now state the main existence result for doubly nonstandard $\mathcal{H}_\infty$ controllers. It should be noted that the Lemma is expressed in terms of two matrices L_F and L_H which must be constructed before the Riccati equations can be solved to check for the existence of controllers. The choice of these matrices is governed by the right half plane invariant zeros of $G_{12}(s)$ and $G_{21}(s)$. Section 3.1 of Chapter 3 contains a discussion of how it is possible to calculate the invariant zeros of the state space realizations of $G_{12}(s)$ and $G_{21}(s)$, as given in (4.1). The matrices L_F and L_H may be constructed using the canonical forms which are introduced in in section 3.1. A discussion of exactly how L_F and L_H may be calculated is presented after the lemma statement.

Lemma 4.1.2 *Suppose one is given a generalized plant $G(s)$ with realization (4.1) satisfying assumptions* **A.1**, **A.2** *and* **A.3** *and with both D_{12} and D_{21} nonstandard, i.e. $n_z < n_u$ and $n_w < n_y$. Let $D_{12}{}^{\dagger}, D_{12}{}^{\perp}, D_{21}{}^{\dagger}$ and $D_{21}{}^{\perp}$ be defined as in (4.8) and (4.9). Define*

$$A_{ZF} = A - B_2 D_{12}{}^{\dagger} C_1 + B_2 D_{12}{}^{\perp} L_F \tag{4.10}$$

with L_F chosen such that it stabilizes the controllable modes of the pair

$$(A - B_2 D_{12}{}^{\dagger} C_1, B_2 D_{12}{}^{\perp}). \tag{4.11}$$

Similarly, define

$$A_{ZH} = A - B_1 D_{21}{}^{\dagger} C_2 + L_H D_{21}{}^{\perp} C_2 \tag{4.12}$$

with L_H chosen such that it stabilizes the observable modes of the pair

$$(D_{21}{}^{\perp}C_2, A - B_1D_{21}{}^{\dagger}C_2). \tag{4.13}$$

A necessary and sufficient condition for the existence of a γ-admissible controller for $G(s)$, is that there exist stabilizing solutions $X, Y \geq 0$ of the following algebraic Riccati equations:

$$XA_{ZF} + A_{ZF}{}^TX + X[B_1B_1{}^T - B_2D_{12}{}^{\dagger}(D_{12}{}^{\dagger})^TB_2{}^T]X = 0, \tag{4.14}$$

$$YA_{ZH}{}^T + A_{ZH}Y + Y[C_1{}^TC_1 - C_2{}^T(D_{21}{}^{\dagger})^TD_{21}{}^{\dagger}C_2]Y = 0, \tag{4.15}$$

which satisfy the coupling condition

$$\rho(XY) < 1. \tag{4.16}$$

By stabilizing solutions of (4.14) and (4.15), we mean ones which ensure that the eigenvalues of the following matrices are all in the open left half plane

$$\begin{aligned} A_X &= A_{ZF} + (B_1B_1^T - B_2D_{12}{}^{\dagger}(D_{12}{}^{\dagger})^TB_2^T)X, & (4.17)\\ A_Y &= A_{ZH} + Y(C_1^TC_1 - C_2^T(D_{21}{}^{\dagger})^TD_{21}{}^{\dagger}C_2). & (4.18) \end{aligned}$$

When they exist, stabilizing solutions of the above Riccati equations are independent of the particular choice of L_F and L_H, provided these matrices are chosen as described above.

Let V_F be a full column rank matrix with column space corresponding to the controllable subspace of the pair $(A - B_2D_{12}{}^{\dagger}C_1, B_2D_{12}{}^{\perp})$ and let V_H be a full row rank matrix with row space corresponding to the observable subspace of the pair $(D_{21}{}^{\perp}C_2, A - B_1D_{21}{}^{\dagger}C_2)$. Then it follows that stabilizing solutions of the above Riccati equations satisfy

$$XV_F = 0 \quad \text{and} \quad V_HY = 0. \tag{4.19}$$

Construction of the Matrices L_F and L_H in the Controller Existence Conditions.

We now consider how the matrices L_F and L_H can be constructed as described in the above lemma statement. The construction of such matrices is an important step in checking the controller existence conditions. As will become apparent later, these matrices also appear in the state-space construction of the parametrization of all doubly nonstandard $\mathcal{H}_\infty$ controllers.

We construct L_F and L_H with reference to a controllability canonical form for $(A - B_2D_{12}{}^{\dagger}C_1, B_2D_{12}{}^{\perp})$ and an observability canonical form for $(D_{21}{}^{\perp}C_2, A - B_1D_{21}{}^{\dagger}C_2)$, respectively. In section 3.1 of Chapter 3 it was shown how such canonical forms may be constructed. For convenience, we now recount these results.

Construction of the Matrix L_F.

With D_{12} nonstandard, the invariant zeros of $G_{12}(s)$ are the uncontrollable modes of $(A - B_2 D_{12}{}^{\dagger} C_1, B_2 D_{12}{}^{\perp})$. Let V_F denote a matrix of full column rank whose column space is the controllable subspace of this pair. Let U_F be any full rank matrix whose column space is complementary to that of V_F. Then the matrix

$$T = \begin{pmatrix} U_F & V_F \end{pmatrix} \tag{4.20}$$

is a square, invertible state-space basis transformation which results in the controllability canonical form:

$$T^{-1}(A - B_2 D_{12}{}^{\dagger} C_1)T = \begin{pmatrix} A_0 & 0 \\ A_{01} & A_1 \end{pmatrix}, \tag{4.21}$$

$$T^{-1} B_2 D_{12}{}^{\perp} = \begin{pmatrix} 0 \\ \beta_F \end{pmatrix}. \tag{4.22}$$

Here the invariant zeros of $G_{12}(s)$ are the eigenvalues of A_0 and the pair $(A_1,\ \beta_F)$ is controllable.

Consider now how a matrix L_F may be constructed which ensures the stability of the subset of eigenvalues of $A_{ZF} = A - B_2 D_{12}{}^{\dagger} C_1 + B_2 D_{12}{}^{\perp} L_F$ which correspond to the controllable subspace of the pair $(A - B_2 D_{12}{}^{\dagger} C_1, B_2 D_{12}{}^{\perp})$: without loss of generality, observe that any L_F can be described as

$$L_F = (L_{F1} \;\; L_{F2})T^{-1}, \tag{4.23}$$

and observe that in the new basis,

$$T^{-1} A_{ZF} T = \begin{pmatrix} A_0 & 0 \\ A_{01} + \beta_F L_{F1} & A_1 + \beta_F L_{F2} \end{pmatrix}. \tag{4.24}$$

Hence with L_{F2} chosen such that $\alpha_F = A_1 + \beta_F L_{F2}$ is stable and with L_{F1} arbitrarily chosen, the stability of the modes of A_{ZF} corresponding to the controllable subspace is guaranteed.

Construction of the Matrix L_H.

With D_{21} nonstandard, the invariant zeros of $G_{21}(s)$ correspond to the unobservable modes of the pair $(D_{21}{}^{\perp} C_2, A - B_1 D_{21}{}^{\dagger} C_2)$. Let V_H denote a matrix of full row rank whose row space is the observable subspace of this pair. Let U_H be any full row rank matrix whose row space is complementary to that of V_H. Then the matrix

$$U = \begin{pmatrix} U_H \\ V_H \end{pmatrix} \tag{4.25}$$

is a square, invertible state-space basis transformation which results in the following observability canonical form:

$$U(A - B_1 D_{21}{}^{\dagger} C_2)U^{-1} = \begin{pmatrix} \alpha_0 & \alpha_{10} \\ 0 & \alpha_1 \end{pmatrix} \tag{4.26}$$

$$D_{21}{}^{\perp} C_2 U^{-1} = \begin{pmatrix} 0 & \beta_H \end{pmatrix}. \tag{4.27}$$

Here the invariant zeros of $G_{21}(s)$ are the eigenvalues of α_0 and the pair (β_H, α_1) is observable.

Consider now how a matrix L_H may be constructed which ensures the stability of the subset of eigenvalues of $A_{ZH} = A - B_1 D_{21}{}^{\dagger} C_2 + L_H D_{21}{}^{\perp} C_2$ which correspond to the observable subspace of the pair $(D_{21}{}^{\perp} C_2, A - B_1 D_{21}{}^{\dagger} C_2)$: without loss of generality, observe that any L_H can be described as

$$L_H = U^{-1} \begin{pmatrix} L_{H1} \\ L_{H2} \end{pmatrix}, \tag{4.28}$$

and observe that in the new basis,

$$U A_{ZH} U^{-1} = \begin{pmatrix} \alpha_0 & \alpha_{10} + L_{H1}\beta_H \\ 0 & \alpha_1 + L_{H2}\beta_H \end{pmatrix}. \tag{4.29}$$

Hence with L_{H2} chosen such that $\alpha_H = \alpha_1 + L_{H2}\beta_H$ is stable and with L_{H1} arbitrarily chosen, the stability of the modes of A_{ZH} corresponding to the observable subspace is guaranteed.

4.1.3 A Re-expression of the Singular Existence Results for Nonstandard Plants.

Before proceeding, we re-express the conditions *1–4* of Lemma 4.1.1 for the case of doubly nonstandard plants in terms of Riccati equations instead of Riccati inequalities. The following Lemma constitutes an intermediate step in the proof of the existence conditions.

Lemma 4.1.3 *Given a generalized plant $G(s)$ with state space realization (4.1) satisfying assumptions* **A.1, A.2** *and* **A.3,** *the conditions of Lemma 4.1.1 may be reexpressed as follows:*

- *A solution $X \geq 0$ of the quadratic matrix inequality $F(X) \geq 0$ satisfies conditions 1 and 3 of Lemma 4.1.1 if and only if X is a solution of the algebraic Riccati equation*

$$(A - B_2 D_{12}{}^{\dagger} C_1)^T X + X(A - B_2 D_{12}{}^{\dagger} C_1) + X(B_1 B_1^T - B_2 D_{12}{}^{\dagger} (D_{12}{}^{\dagger})^T B_2^T) X = 0, \tag{4.30}$$

  X also satisfies

$$X B_2 D_{12}{}^{\perp} = 0, \tag{4.31}$$

  and results in the following pair being stabilizable

$$\left(A - B_2 D_{12}{}^{\dagger} C_1 + B_1 B_1^T X - B_2 D_{12}{}^{\dagger} (D_{12}{}^{\dagger})^T B_2^T X,\ B_2 D_{12}{}^{\perp}\right). \tag{4.32}$$

- *A solution $Y \geq 0$ of the quadratic matrix inequality $H(Y) \geq 0$ satisfies conditions 2 and 4 of Lemma 4.1.1 if and only if Y is a solution of the algebraic Riccati* *equation*

$$Y(A - B_1D_{21}{}^{\dagger}C_2)^T + (A - B_1D_{21}{}^{\dagger}C_2)Y + Y(C_1{}^TC_1 - C_2{}^T(D_{21}{}^{\dagger})^TD_{21}{}^{\dagger}C_2)Y = 0, \tag{4.33}$$

Y also satisfies

$$D_{21}{}^{\perp}C_2Y = 0 \tag{4.34}$$

and results in the following pair being detectable

$$\left(D_{21}{}^{\perp}C_2,\ A - B_1D_{21}{}^{\dagger}C_2 + YC_1^TC_1 - YC_2^T(D_{21}{}^{\dagger})^TD_{21}{}^{\dagger}C_2\right). \tag{4.35}$$

Proof: Condition *1* of Lemma 4.1.1 says that with $X \geq 0$ such that $F(X) \geq 0$, $\text{rank}\{F(X)\} = \text{normrank}\{G_{12}(s)\}$. Since we assume here that D_{12} is nonstandard and full row rank, this means that $\text{normrank}\{G_{12}(s)\} = \text{rank}\{D_{12}\} = n_z$.

Observe now that for any invertible matrix M, $\text{rank}\{F(X)\} = \text{rank}\{M^TF(X)M\}$. Also observe that for such an M, $F(X) \geq 0$ if and only if $M^TF(X)M \geq 0$. Next define the following invertible matrix

$$J = \begin{pmatrix} I_n & 0 & 0 \\ 0 & D_{12}{}^{\dagger} & D_{12}{}^{\perp} \end{pmatrix} \begin{pmatrix} I_n & 0 & 0 \\ -\left((D_{12}{}^{\dagger})^TB_2^TX + C_1\right) & I_{n_z} & 0 \\ 0 & 0 & I_{n_u - n_z} \end{pmatrix}. \tag{4.36}$$

Straightforward algebra can be employed to verify that

$$J^TF(X)J = \begin{pmatrix} \mathcal{R}(X) & 0 & XB_2D_{12}{}^{\perp} \\ 0 & I_{n_z} & 0 \\ (D_{12}{}^{\perp})^TB_2^TX & 0 & 0_{(n_u-n_z)\times(n_u-n_z)} \end{pmatrix}, \tag{4.37}$$

with the definition:

$$\mathcal{R}(X) = (A - B_2D_{12}{}^{\dagger}C_1)^TX + X(A - B_2D_{12}{}^{\dagger}C_1) + X(B_1B_1^T - B_2D_{12}{}^{\dagger}(D_{12}{}^{\dagger})^TB_2^T)X. \tag{4.38}$$

The matrix given in (4.37) is nonnegative definite and has rank n_z *if and only if* $\mathcal{R}(X) = 0$, i.e.

$$(A - B_2D_{12}{}^{\dagger}C_1)^TX + X(A - B_2D_{12}{}^{\dagger}C_1) + X(B_1B_1^T - B_2D_{12}{}^{\dagger}(D_{12}{}^{\dagger})^TB_2^T)X = 0, \tag{4.39}$$

and simultaneously

$$XB_2D_{12}{}^{\perp} = 0. \tag{4.40}$$

Condition *3* of Lemma 4.1.1 says that when $\Re\{s\} \geq 0$,

$$\text{rank}\begin{pmatrix} L(X,s) \\ F(X) \end{pmatrix} = n + \text{normrank}\{G_{12}(s)\} = n + n_z. \tag{4.41}$$

Note that for any invertible matrices M and N having appropriate dimensions,

$$\text{rank}\left\{N\begin{pmatrix} L(X,s) \\ F(X) \end{pmatrix}M\right\} = \text{rank}\begin{pmatrix} L(X,s) \\ F(X) \end{pmatrix}. \tag{4.42}$$

Provided condition *1* of Lemma 4.1.1 holds, i.e. (4.39) and (4.40) are true for some $X \geq 0$, it can again be verified by straightforward algebra that

$$\begin{pmatrix} I_n & 0 \\ 0 & J^T \end{pmatrix}\begin{pmatrix} L(X,s) \\ F(X) \end{pmatrix} J =$$
$$\begin{pmatrix} sI - A - B_1B_1^TX + B_2D_{12}{}^\dagger(D_{12}{}^\dagger)^TB_2^TX + B_2D_{12}{}^\dagger C_1 & 0 & -B_2D_{12}{}^\perp \\ 0_{n\times n} & 0 & 0 \\ 0 & I_{n_z} & 0 \\ 0 & 0 & 0_{(n_u-n_z)\times(n_u-n_z)} \end{pmatrix}.$$

Observe that the rank condition in (4.41) is equivalent to the above expression having rank $n + n_z$ in the closed right half plane. Hence provided (4.39) and (4.40) hold, an equivalent statement of condition *3* of Lemma 4.1.1 is that the following matrix has full row rank in the closed right half plane:

$$\begin{pmatrix} sI - A - B_1B_1^TX + B_2D_{12}{}^\dagger(D_{12}{}^\dagger)^TB_2^TX + B_2D_{12}{}^\dagger C_1 & -B_2D_{12}{}^\perp \end{pmatrix}, \tag{4.43}$$

or equivalently that the following is a stabilizable pair:

$$\left(A - B_2D_{12}{}^\dagger C_1 + B_1B_1^TX - B_2D_{12}{}^\dagger(D_{12}{}^\dagger)^TB_2^TX,\ B_2D_{12}{}^\perp\right). \tag{4.44}$$

Condition *1* of Lemma 4.1.1 says that with $X \geq 0$ such that $F(X) \geq 0$, $\mathrm{rank}\{F(X)\} = \mathrm{normrank}\{G_{12}(s)\}$. Since we assume here that D_{12} is nonstandard and full row rank, this means that $\mathrm{normrank}\{G_{12}(s)\} = \mathrm{rank}\{D_{12}\} = n_z$.

Analogous reasoning to the above can be used to establish that a matrix $Y \geq 0$ such that $H(Y) \geq 0$ satisfies conditions *2* and *4* of Lemma 4.1.1 if and only if Y is a nonnegative definite solution of the ARE (4.33) which satisfies (4.34) and for which the pair (4.35) is detectable. □

4.1.4 Proof of the Nonstandard Existence Conditions.

Since **A.1** and **A.3** hold for the plants considered here, the existence conditions presented in Lemma 4.1.1 of subsection 4.1.1 apply. With this in mind, we shall prove the necessity and sufficiency of the nonstandard existence conditions given in Lemma 4.1.2. These proofs draws upon the re-expression of the singular existence conditions presented in the previous subsection.

Proof of Necessity.

If an $\mathcal{H}_\infty$ controller exists for (4.1), then matrices $X \geq 0$ and $Y \geq 0$ exist which satisfy $F(X) \geq 0$, $H(Y) \geq 0$, $\rho(XY) < 1$ and which ensure satisfaction of the conditions *1–4* in Lemma 4.1.1. Using Lemma 4.1.3, we shall now show that the existence conditions stated in Lemma 4.1.2 follow as a necessary consequence of the above facts.

From Lemma 4.1.3, $X \geq 0$ as defined above satisfies the ARE (4.39), the identity (4.40) and ensures that the pair (4.44) is stabilizable. We now show that this also guarantees that X solves the ARE (4.14).

If we right-multiply the ARE (4.39) by the matrix $B_2 D_{12}{}^{\perp}$ and use the identity (4.40), we obtain $X(A - B_2 D_{12}{}^{\dagger} C_1) B_2 D_{12}{}^{\perp} = 0$. More generally, an inductive argument is now given which shows that the following identity holds for all nonnegative integers i:

$$X(A - B_2 D_{12}^{\dagger} C_1)^i B_2 D_{12}^{\perp} = 0. \tag{4.45}$$

Suppose that (4.45) holds for some nonnegative integer i. If we then right-multiply (4.39) by the matrix $(A - B_2 D_{12}^{\dagger} C_1)^i B_2 D_{12}^{\perp}$, then it follows that $X(A - B_2 D_{12}^{\dagger} C_1)^{i+1} B_2 D_{12}^{\perp} = 0$. Thus the row space of X is perpendicular to the controllable subspace of the pair

$$\left(A - B_2 D_{12}^{\dagger} C_1, B_2 D_{12}^{\perp}\right). \tag{4.46}$$

It follows immediately that $XV_F = 0$ since the column space of V_F by definition spans the controllable subspace of the pair $(A - B_2 D_{12}{}^{\dagger} C_1, B_2 D_{12}{}^{\perp})$. If one now introduces the basis transformation T as defined in (4.20), corresponding to the canonical form in (4.21) and (4.22), it follows that

$$T^T X T = \begin{pmatrix} \Psi & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.47}$$

with partitioning conformal to that in the canonical form and Ψ nonzero in general.

Since the pair (4.44) is stabilizable, it follows that there exists a matrix $\tilde{L} \in \mathbb{R}^{(n_u - n_z)\times n}$ such the following matrix is stable

$$A - B_2 D_{12}{}^{\dagger} C_1 + B_1 B_1^T X - B_2 D_{12}{}^{\dagger} (D_{12}{}^{\dagger})^T B_2^T X + B_2 D_{12}{}^{\perp} \tilde{L}. \tag{4.48}$$

Note that any matrix $\tilde{L}$ as given in (4.48) can, without loss of generality, be expressed as

$$\tilde{L} = (\tilde{L}_1 \ \ \tilde{L}_2) T^{-1}, \tag{4.49}$$

where $\tilde{L}_1$ and $\tilde{L}_2$ have the same number of columns as A_0 and A_1, respectively. It can be verified that, as a consequence of the structure of X in (4.47) and the controllability canonical form in (4.21) and (4.22), that when expressed in the new basis, the stable matrix (4.48) reads

$$\begin{pmatrix} \times & 0 \\ \times & A_1 + \beta_F \tilde{L}_2 \end{pmatrix}. \tag{4.50}$$

Here $\times$ denotes entries whose exact values are inconsequential to our argument. Since the above matrix is by hypothesis stable, its $(1,1)$ block must also be stable. It is fairly straightforward to show that the $(1,1)$ block is also independent of $\tilde{L}$. Another consequence of the stability of (4.50) is that $A_1 + \beta_F \tilde{L}_2$ is stable. It follows that an appropriate choice for $\tilde{L}$ is $\tilde{L} = L_F$, with L_F as described in the statement of Lemma 4.1.2.

It follows from (4.39) and (4.40) that $X \geq 0$ solves the ARE (4.14) and the comments immediately above establish that it is also a stabilizing solution of that equation.

An identical argument establishes the necessity of the existence of a stabilizing solution

$Y \geq 0$ of (4.15). The coupling condition (4.16) follows directly from Lemma 4.1.1.

Proof of Sufficiency.

Suppose (4.14) has a stabilizing solution $X \geq 0$. Suppose that L_F has been chosen as described in the Lemma statement, such that it stabilizes the pair $(A - B_2 D_{12}{}^{\dagger} C_1, B_2 D_{12}{}^{\perp})$.

Recall that L_F has the structure $L_F = (L_{F1} \;\; L_{F2})T^{-1}$ where the matrix L_{F2} is chosen such that it stabilizes $\alpha_F = A_1 + \beta_F L_{F2}$.

Note that (4.14) can be rewritten as

$$XA_{ZF} + A_X^T X = 0, \tag{4.51}$$

where $A_{ZF} = A - B_2 D_{12}{}^{\dagger} C_1 + B_2 D_{12}{}^{\perp} L_F$ and $A_X = A_{ZF} + (B_1 B_1^T - B_2 D_{12}{}^{\dagger} (D_{12}{}^{\dagger})^T B_2^T) X$. If one now introduces the basis transformation defined in (4.20), one obtains

$$T^{-1} A_{ZF} T = \begin{pmatrix} A_0 & 0 \\ A_{01} + \beta_F L_{F1} & \alpha_F \end{pmatrix} \tag{4.52}$$

and it follows that $A_{ZF} V_F = V_F \alpha_F$, where V_F is a full column rank matrix whose column space is the controllable subspace of the pair $(A - B_2 D_{12}{}^{\dagger} C_1, B_2 D_{12}{}^{\perp})$. Hence, right-multiplying (4.51) by V_F results in the identity $XV_F \alpha_F + A_X^T X V_F = 0$.. Since both α_F and A_X are by hypothesis stable, one can apply the Lemma of Lyapunov, (see Lemma 0.0.1 in the summary of **Notation, Definitions and Fundamental Results** at the beginning of this thesis) directly to the above equation to deduce that $XV_F = 0$. Recall that V_F is full column rank. It follows that X has the structure given in (4.47). A consequence of this and the identity (4.22) is that $XB_2 D_{12}{}^{\perp} = 0$ and also that $XA_{ZF} = X(A - B_2 D_{12}{}^{\dagger} C_1)$, from which it follows that X, in addition to being a solution of (4.14), is also a solution of (4.39).

Next observe that the matrix A_X results when one applies the state feedback matrix L_F to the pair (4.44). Since A_X is stable by hypothesis, it follows that (4.44) must be a stabilizable pair.

In summary, we have established the existence of a real symmetric matrix $X \geq 0$ which solves (4.39), satisfies (4.40) and ensures that the pair (4.44) is stabilizable. It follows from Lemma 4.1.3 that items *1* and *3* of Lemma 4.1.1 hold.

Given a stabilizing solution $Y \geq 0$ of (4.15), analogous arguments to the above establish conditions *2* and *4* of Lemma 4.1.1.

The coupling condition $\rho(XY) < \gamma^2$ holds trivially by hypothesis. Each of the conditions of Lemma 4.1.1 therefore hold, from which it follows that there exists at least one doubly nonstandard $\mathcal{H}_\infty$ controller. □

4.2 The Nonstandard State Feedback Problem.

In this section, we lay foundations for the full parametrization of nonstandard output feedback controllers by developing a solution for the state-feedback case. This work draws directly on the existence conditions established in the previous section. The existence of a constant $\mathcal{H}_\infty$ state-feedback gain matrix is established for a state feedback problem associated with the realization of the generalized plant given in (4.1). The state-feedback law is constructed using the solution X of the ARE (4.14) given in Lemma 4.1.2.

The Generalized Plant for the State Feedback Problem.

Consider the following state-feedback problem which is derived from the original doubly nonstandard problem, with the only difference being that all states are measureable and uncorrupted by the signal $w(t)$; i.e. $y(t) = x(t)$. The associated generalized plant reads:

$$G^{SF}(s) = \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ I_n & 0 & 0 \end{array} \right). \tag{4.53}$$

It should be noted that the matrix D_{12} is assumed to be nonstandard (i.e. $n_z < n_u$). It is generally the case that $n_w < n = n_y$, with the result that D_{21} is nonstandard. Note that assumption **A.2** is also violated since $G_{21}^{SF}(\infty) = 0$. Thus using standard results alone we cannot present a *complete* parametrization of all (possibly dynamic) state-feedback laws. However, with the existence results from section 4.1, it is shown in the present section that a constant state-feedback control strategy is available if there exists a nonnegative definite stabilizing solution X of the ARE (4.14).

A Special Structure for Constant State Feedback Laws.

Note first that *any* state-feedback law F for the pair (A, B_2) can be written *without loss of generality* as:

$$F = -D_{12}{}^{\dagger}C_1 + D_{12}{}^{\perp}L_F + D_{12}{}^{\dagger}E_F, \tag{4.54}$$

where

$$\begin{pmatrix} E_F \\ L_F \end{pmatrix} = \begin{pmatrix} C_1 \\ 0 \end{pmatrix} + \begin{pmatrix} D_{12} \\ (D_{12}{}^{\perp})^T \end{pmatrix} F. \tag{4.55}$$

Remark: Note that the first term in this state-feedback law, if implemented *on its own* results in *complete disturbance decoupling*, in the sense that $z(t) = 0$. However, the closed loop state dynamics matrix $A - B_2D_{12}{}^{\dagger}C_1$ which result from this is *not* necessarily stable. In fact, if $G_{12}(s)$ has any right half plane invariant zeros, these become closed loop modes. This can be easily gleaned from the controllability canonical form for $(A - B_2D_{12}{}^{\dagger}C_1, B_2D_{12}{}^{\perp})$ which was introduced earlier. □

Consider the closed-loop state dynamics matrix which results from implementing

(4.54):

$$A + B_2F \;=\; A - B_2{D_{12}}^{\dagger}C_1 + B_2{D_{12}}^{\perp}L_F + B_2{D_{12}}^{\dagger}E_F. \tag{4.56}$$

Observe now that one can introduce a basis transformation of the form described in (4.20) and apply the identities (4.21) and (4.22) to obtain:

$$T^{-1}(A + B_2F)T \;=\; \begin{pmatrix} A_0 & 0 \\ A_{01} & A_1 \end{pmatrix} + \begin{pmatrix} 0 \\ \beta_F \end{pmatrix} L_F + \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} E_F, \tag{4.57}$$

where, for convenience, we have partitioned the matrix $B_2{D_{12}}^{\dagger}$ in this basis as follows:

$$T^{-1}B_2{D_{12}}^{\dagger} = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}. \tag{4.58}$$

Consider the family of state-feedback matrices which arises when one makes the following particular choices of L_F and E_F:

$$L_F \;=\; (L_{F1} \;\; L_{F2})T^{-1}, \tag{4.59}$$

$$E_F \;=\; (E_1 \;\; 0)T^{-1}, \tag{4.60}$$

where L_{F1} and E_1 have the same number of columns as A_0, and L_{F2} has the same number of columns as A_1. Since (A_1, β_F) is controllable, it is possible to choose that L_{F2} such that $\alpha_F = A_1 + \beta_F L_{F2}$ is stable. The structure of E_F given in (4.60) ensures that $T^{-1}(A + B_2F)T$ has the following block triangular structure:

$$T^{-1}A_FT = \begin{pmatrix} A_0 + \gamma_1E_1 & 0 \\ A_{01} + \beta_F L_{F1} + \gamma_2E_1 & A_1 + \beta_F L_{F2} \end{pmatrix}. \tag{4.61}$$

Suppose next that the partition E_1 is chosen such that it stabilizes $A_0 + \gamma_1E_1$. The effect of such a choice of E_1 on A_F is to move the right half plane modes of A_0 (corresponding to right half plane zeros of $G_{12}(s)$) into the left half plane. The result is that the closed loop matrix $A_F = A + B_2F$ is stable.

Note that it is always possible to choose the partition E_1 such that it stabilizes $A_0 + \gamma_1E_1$. This follows from the following reasoning: observe that stabilizability of (A, B_2) is equivalent to the stabilizability of $(A - B_2{D_{12}}^{\dagger}C_1, B_2)$ which in turn is equivalent to the stabilizability of $(A - B_2{D_{12}}^{\dagger}C_1, (B_2{D_{12}}^{\dagger} \;\; B_2{D_{21}}^{\perp}))$. Applying the change of basis T to this latter pair results in the pair

$$\left(\begin{pmatrix} A_0 & 0 \\ A_{01} & A_1 \end{pmatrix}, \begin{pmatrix} \gamma_1 & 0 \\ \gamma_2 & \beta_F \end{pmatrix} \right). \tag{4.62}$$

With L_{F2} chosen such that $\alpha_F = A_1 + \beta_F L_{F2}$ is stable, application of the feedback transformation

$$\begin{pmatrix} 0 & 0 \\ 0 & E_1 \end{pmatrix} \tag{4.63}$$

to the pair 4.62 allows us to conclude that the following pair is stabilizable:

$$\left(\begin{pmatrix} A_0 & 0 \\ A_{01} & A_1 + \beta_F E_1 \end{pmatrix}, \begin{pmatrix} \gamma_1 & 0 \\ \gamma_2 & \beta_F \end{pmatrix} \right). \tag{4.64}$$

This implies the required stabilizability of (A_0, γ_1). (This can be checked using the standard rank test for stabilizability which is given at the beginning of this thesis in the section **Notation, Definitions and Fundamental Results.**)

Closed Loop Transfer Function with the Special State Feedback Law.

Consider the closed loop transfer function which results when a state-feedback law F is implemented:

$$G_{zw}^{SF} = (C_1 + D_{12}F)(sI - (A + B_2F))^{-1}B_1. \tag{4.65}$$

Suppose in addition that F has the special structure described above. It follows then that

$$G_{zw}^{SF} = E_F(sI - A_F)^{-1}B_1. \tag{4.66}$$

Applying the basis transformation T and noting the special structure of E_F and A_F in that basis reveals that

$$G_{zw}^{SF} = \begin{pmatrix} E_1(sI - (A_0 + \gamma_1 E_1))^{-1} & 0 \end{pmatrix} T^{-1}B_1. \tag{4.67}$$

This has the important consequence that the closed loop transfer function is actually *independent* of the closed loop modes associated with $A_1 + \beta_F L_{F2}$. Note also that this closed-loop transfer function is independent of the matrix L_F. However, L_F must be chosen as described above to ensure closed loop internal stability.

A Nonstandard $\mathcal{H}_\infty$ State Feedback Law via the Riccati Equation.

It should be emphasized that the above observations are made independently of any consideration of the closed-loop $\mathcal{H}_\infty$ objective. However, we now examine a particular choice of F which has the structure described above and which is of importance in the context of $\mathcal{H}_\infty$ control.

Lemma 4.2.1 *Suppose one is given a generalized plant $G(s)$ with realization (4.53) satisfying assumptions* **A.1, A.2** *and* **A.3** *with D_{12} nonstandard, i.e. $n_z < n_u$. Suppose a nonnegative definite stabilizing solution X of (4.14) has been found.*

1. *With the definition $E_F = -(D_{12}{}^\dagger)^T B_2^T X$, this matrix has the particular structure described in (4.60).*

2. *With the above definition of E_F, let F_∞ be defined according to (4.54):*

$$F = -D_{12}{}^\dagger C_1 + D_{12}{}^\perp L_F - D_{12}{}^\dagger (D_{12}{}^\dagger)^T B_2^T X, \tag{4.68}$$

   where L_F stabilizes the controllable subspace of the pair $(A - B_2 D_{12}{}^\dagger C_1, B_2 D_{12}{}^\perp)$. Then the matrix $A_\infty = A + B_2 F_\infty$ is stable.

3. *Moreover, with the above choice of state-feedback control, the resulting closed loop transfer function matrix is bounded real:*

$$\|(C_1 + D_{12}F_\infty)(sI - (A + B_2F_\infty))^{-1}B_1\|_\infty < 1. \tag{4.69}$$

Proof:

1. From the structure of T^TXT established in (4.47) and the structure of $T^{-1}B_2D_{12}{}^\dagger$ established in (4.22), it follows that

$$E_F = (-\gamma_1^T\Psi \;\; 0)T^{-1}. \tag{4.70}$$

2. Note from (4.14) that

$$XA_\infty + A_\infty^TX + X(B_1B_1^T + B_2D_{12}{}^\dagger(D_{12}{}^\dagger)^TB_2^T)X = 0. \tag{4.71}$$

Since $A_X = A_\infty + B_1B_1^TX$ is, by hypothesis, stable, it follows that (B_1^TX, A_∞) is a detectable pair. Since $X \geq 0$, it follows immediately from (4.71) and standard Lyapunov stability results that A_∞ is stable.

3. To show the identity (4.69), we apply the bounded real lemma to the closed loop system. Note that $\tilde{C}_1 + D_{12}F_\infty = E_F$ and therefore that the closed loop system can be written as $E_F(sI-(A+B_2F_\infty))^{-1}B_1$. Next note that one can re-write (4.71) as follows:

$$X(A+B_2F_\infty) + (A+B_2F_\infty)^TX + XB_1B_1^TX + E_F^TE_F = 0. \tag{4.72}$$

Observe that (4.72) is in fact the ARE which arises when the bounded real lemma is applied to the closed loop system. $X \geq 0$ is a stabilizing solution of this equation. The (strictly) bounded real lemma (see [85]) states that the existence of such a solution to (4.72) is a necessary and sufficient condition for satisfaction of the bound given in (4.69). □

4.3 A Re-expression of the Nonstandard Controller Synthesis Problem.

This section describes the first step in the derivation of the parametrization of all case 2 controllers. The derivation continues in subsequent sections and finally the result is presented in Theorem 4.5.2 in section 4.5. We do not derive the singly nonstandard results for case 1 and case 3 plants, which is presented in Theorems 4.5.1 and 4.5.3 respectively, since proofs for these cases follow along similar lines to, and are in fact considerably simpler than, the proof for the case 2 problem.

The derivation is based on the existence conditions and state-feedback control law which were established in sections 4.1 and 4.2 respectively. We assume throughout the existence of at least one case 2 $\mathcal{H}_\infty$ controller and thus the existence of matrices X and Y satisfying the conditions given in Lemma 4.1.2.

Before proceeding, we give an outline of the approach taken in proving the controller parametrization result. In subsection 4.3.1, we review some well known results from linear systems theory which are used in the controller derivation. Next, it is shown in subsection 4.3.2 that the state-feedback law derived in section 4.2 can be used to decompose the generalized plant into two components. It is shown that $G(s)$ is expressible as a linear fractional map of a lossless system $\Theta(s)$ and another system, the so-called *temporary* generalized plant $G_{tmp}(s)$. A similar decomposition was employed as part of the solution to the standard $\mathcal{H}_\infty$ problem in [25]. It will become evident that one can *equivalently* and *more easily* treat the $\mathcal{H}_\infty$ problem for the temporary generalized plant than the $\mathcal{H}_\infty$ problem for the original generalized plant.

Direct application of standard results to $G_{tmp}(s)$ is, however not possible. In order to find a full parametrization of $\mathcal{H}_\infty$ controllers , our strategy in subsection 4.3.3 is to first describe all stabilizing controllers for $G_{tmp}(s)$. In this step, the state-space Youla parametrization presented in Lemma 4.3.1 is applied to $G_{tmp}(s)$, resulting in a parametrization of stabilizing controllers as a function of a stable free parameter $Q(s)$. As will become apparent in subsection 4.3.3, one important advantage in treating $G_{tmp}(s)$ is that the set of all internally stable closed-loop systems can be described using a linear fractional transformation of the free stable parameter $Q(s)$ with a coefficient matrix which has state state dimension only that of $G_{tmp}(s)$ (and thus of $G(s)$), rather than twice this dimension which is generally the case. By writing down the corresponding set of all internally stable closed-loop transfer function matrices, one can pose a new $\mathcal{H}_\infty$ control problem in which the Youla parameter $Q(s)$ is required to be chosen as a control law. In section 4.4, it is shown that the $\mathcal{H}_\infty$ problem defined at the end of section 4.3 can be expressed in terms of a standard *output estimation* $\mathcal{H}_\infty$ controller synthesis problem.

4.3.1 Preliminaries.

We now summarize a number of facts which find application in the controller derivation which follows. Note that these facts are general observations and are not specific to the $\mathcal{H}_\infty$ objective.

Youla Parametrization of all Stabilizing Controllers.

In order to fully investigate the freedom in nonstandard $\mathcal{H}_\infty$ control laws, we will later employ the Youla parametrization of all stabilizing controllers, which we now recount. Note that this result is applicable to both standard and nonstandard plants.

Lemma 4.3.1 *Let $G(s)$ be a generalized plant as given in (4.1) which satisfies* **A.1**. *Let F and H be any matrices which stabilize*

$$A_F = A + B_2F \quad \text{and} \quad A_H = A + HC_2 \quad \text{respectively.} \tag{4.73}$$

The set of all controllers $K(s)$ which internally stabilize $G(s)$ is described by

$$K(s) = LFT\left\{\left(\begin{array}{c|cc} A + B_2F + HC_2 & -H & -B_2 \\ \hline F & 0 & -I_{n_u} \\ -C_2 & I_{n_y} & 0 \end{array}\right), Q(s)\right\} \tag{4.74}$$

where $Q(s) \in \mathcal{RH}_\infty$ is a free parameter. Moreover, (4.74) defines an invertible mapping *between stabilizing controllers and stable transfer function matrices $Q(s) \in \mathcal{RH}_\infty$.*

The set of all internally stable closed-loop operators $T_{zw}(s)$ can be described in terms of the same parameter $Q(s)$ as

$$T_{zw}(s) = LFT\left\{\left(\begin{array}{cc|cc} A_F & -B_2F & B_1 & B_2 \\ 0 & A_H & B_1+HD_{21} & 0 \\ \hline C_1+D_{12}F & -D_{12}F & 0 & D_{12} \\ 0 & C_2 & D_{21} & 0 \end{array}\right), -Q(s)\right\}. \quad (4.75)$$

Proof: Derivations of these results are presented in [33] and Appendix A of [34]. □

Realizations of different Order for the same Transfer Function Matrix.

The result described in the following lemma will be later used to enable the nonstandard problem to be expressed as a standard problem. It is a straightforward extension of standard state space transformation ideas.

Lemma 4.3.2

1. *Given a transfer function with realization $G(s) = C(sI - A)^{-1}B$, let α, β, θ be any matrices such that $AV = V\alpha$, $B = V\beta$ and $CV = \theta$, where V is of full column rank, then*

$$\left(\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right) = \left(\begin{array}{c|c} \alpha & \beta \\ \hline \theta & 0 \end{array}\right). \quad (4.76)$$

2. *Furthermore, if $\theta = 0$, then the transfer function is identically zero.*

Proof: **1** follows since the Markov parameters of the two systems are identical; for all nonnegative integers i, $CA^iB = CA^iV\beta = CV\alpha^i\beta = \theta\alpha^i\beta$. **2** is then immediate. □

4.3.2 Lossless Decomposition and the Temporary Generalized Plant.

In the solution of the standard output feedback $\mathcal{H}_\infty$ control problem in [25], a lossless decomposition of the generalized plant is introduced as a preliminary step. This allows the $\mathcal{H}_\infty$ synthesis problem to be equivalently expressed in terms of a simpler (output estimation) $\mathcal{H}_\infty$ problem for a so-called *temporary* generalized plant. In [25], a nonnegative definite stabilizing solution of the standard full-information Riccati equation (see equation (4.147) in section 4.5) makes a state-space construction of the components of this decomposition possible. We now demonstrate that a plant having D_{12} nonstandard can be decomposed in a similar manner, given a nonnegative definite stabilizing solution X of

$$XA_{ZF} + A_{ZF}{}^TX + X[B_1B_1{}^T - B_2D_{12}{}^\dagger(D_{12}{}^\dagger)^TB_2{}^T]X = 0. \quad (4.77)$$

Moreover, the nonstandard $\mathcal{H}_\infty$ problem can also be equivalently expressed in terms of an $\mathcal{H}_\infty$ problem for a *temporary* generalized plant $G_{tmp}(s)$. However, whilst being simpler to deal with than the original nonstandard plant, the temporary plant is not directly amenable to application of standard state-space results. It is the subject of later sections as to how the $\mathcal{H}_\infty$ problem for $G_{tmp}(s)$ can be solved.

Lemma 4.3.3 *Suppose there exists a solution to the $\mathcal{H}_\infty$ control problem for a generalized plant $G(s)$ as given in (4.1) which satisfies* **A.1** , **A.2** *and* **A.3** *with D_{12} nonstandard ($n_z < n_u$). Let X be the nonnegative definite stabilizing solution of (4.77), whose existence is guaranteed by Lemma 4.1.2. Define the following matrices:*

$$\hat{A} = A + B_1 B_1^T X, \tag{4.78}$$
$$\hat{C}_2 = C_2 + D_{21} B_1^T X, \tag{4.79}$$
$$E_F = -(D_{12}{}^\dagger)^T B_2^T X. \tag{4.80}$$

Then with $G_{tmp}(s)$ defined as

$$G_{tmp}(s) = \left(\begin{array}{c|cc} \hat{A} & B_1 & B_2 \\ \hline C_1 - E_F & 0 & D_{12} \\ \hat{C}_2 & D_{21} & 0 \end{array}\right), \tag{4.81}$$

$K(s)$ is an $\mathcal{H}_\infty$ controller for $G(s)$ if and only if it is an $\mathcal{H}_\infty$ controller for $G_{tmp}(s)$.

Proof: Consider the following transfer function matrix:

$$\Theta(s) = \left(\begin{array}{c|cc} A + B_2 F_\infty & B_1 & B_2 D_{12}{}^\dagger \\ \hline E_F & 0 & I_{n_z} \\ -B_1{}^T X & I_{n_w} & 0 \end{array}\right) \in \mathcal{RH}_\infty \tag{4.82}$$

It is shown in Appendix D that for any controller $K(s)$,

$$LFT\{\Theta, LFT\{G_{tmp}, K\}\} = LFT\{G, K\} = T_{zw}(s). \tag{4.83}$$

Note in particular that this identity holds without any unstable pole/zero cancellations. As discussed in Appendix D, in deriving the right hand side of (4.83) from the left hand side, it is apparent that some nonminimal modes can be eliminated. These modes are stable and hence do not affect the internal stability of the closed-loop system.

Let $\Theta(s) = D_\theta + C_\theta(sI - A_\theta)^{-1} B_\theta$ denote the realization defined in (4.82). One can confirm that $\Theta(s)$ is *inner*, in other words it satisfies the identity

$$\Theta^T(-s)\Theta(s) = I_{n_u + n_w}. \tag{4.84}$$

This can be established using a well known state-space check of this property (see for example Chapter 3 of [34]): the identity (4.84) holds due to the following equalities

$$\begin{array}{l} X A_\theta + A_\theta{}^T X + C_\theta{}^T C_\theta = 0, \\ D_\theta{}^T C_\theta + B_\theta{}^T X = 0 \quad \text{and} \quad D_\theta{}^T D_\theta = I_{n_z + n_w}. \end{array} \tag{4.85}$$

Here $X \geq 0$ is the stabilizing solution of (4.77). The first of these equalities is simply a re-writing of the ARE (4.71) which was obtained under the same conditions in the proof of Lemma 4.2.1. The second equality is a direct consequence of the value of E_F assumed in the lemma statement. The third equality is easily checked.

Note also that $\Theta_{21}^{-1}(s) \in \mathcal{RH}_\infty$ since the zeros of $\Theta_{21}(s)$ are the eigenvalues of $A + B_2F_\infty + B_1B_1^TX = A_{ZF} + (B_1B_1^T - B_2D_{12}{}^\dagger(D_{12}{}^\dagger)^TB_2^T)X = A_X$ which is, by hypothesis, a stable matrix. Observe also that $\Theta_{22}(\infty) = 0$. Under these conditions on $\Theta(s)$, the following fact holds (see Lemma 15 of [25] and Theorem 4.3.3 of [34]):

Given any transfer function matrix $\Omega(s)$, *then* $\Omega(s) \in \mathcal{BH}_\infty$
if and only if
both $LFT\{\Theta, \Omega\} \in \mathcal{BH}_\infty$ and *this interconnection is internally stable.*

By applying this result with $\Omega(s) = LFT\ \{G_{tmp}(s), K(s)\}$, the stated result follows. □

Thus the set of $\mathcal{H}_\infty$ controllers for $G(s)$ is equivalent to the set of $\mathcal{H}_\infty$ controllers for $G_{tmp}(s)$. Whilst $G_{tmp}(s)$ is also a nonstandard plant, it will be shown subsequently that it can be more easily transformed into a problem which can be solved using standard $\mathcal{H}_\infty$ results, than can $G(s)$ alone. The main objective is now therefore to try and parametrize all $\mathcal{H}_\infty$ controllers for $G_{tmp}(s)$, thereby obtaining those for $G(s)$.

4.3.3 Youla Parametrization for the Temporary Plant.

We now apply the state space Youla parametrization result of Lemma 4.3.1 to the realization of $G_{tmp}(s)$ given in (4.81). Recall that Lemma 4.3.1 can be applied without restriction to nonstandard plants.

Existence of Stabilizing F and H Matrices for the Youla Parametrization.

With reference to Lemma 4.3.1, note that the Youla parametrization constitutes an invertible mapping between internally stabilizing controllers and transfer function matrices $Q(s) \in \mathcal{RH}_\infty$, provided we make a choice of F and H which stabilize the matrices $A + B_2F$ and $A + HC_2$ respectively. It is now argued that provided an $\mathcal{H}_\infty$ controller exists for the original nonstandard generalized plant, a choice of stabilizing F and H matrices is always possible for the realization of $G_{tmp}(s)$ given in (4.81).

With X the nonnegative definite stabilizing solution of (4.77), we choose $F = F_\infty$ as defined in Lemma 4.2.1. With reference to the definition of A_X in (4.17) and the definition of $\hat{A}$ in (4.78), note the following equality:

$$\hat{A} + B_2F_\infty \quad = \quad A_X. \tag{4.86}$$

Since A_X (as given in 4.17) is stable by hypothesis (since X is a stabilizing solution of 4.77), it follows that $F = F_\infty$ is a suitable choice of stabilizing state feedback law for the Youla parametrization of all stabilizing controllers of $G_{tmp}(s)$.

Since we have assumed the existence of an $\mathcal{H}_\infty$ controller for $G(s)$, by Lemma 4.3.3 there must exist an $\mathcal{H}_\infty$ controller for $G_{tmp}(s)$ in (4.81) which, by definition, must be internally stabilizing. Thus $(\hat{A}, B_2, \hat{C}_2)$ is stabilizable and detectable. It follows immediately that there exists at least one matrix H such that $A_H = \hat{A} + H\hat{C}_2$ is stable. (How such an H can be constructed will become evident Lemma 4.4.3.) At the present stage in the argument, we need only know that it exists.

Parametrization of Controllers and Closed Loops for the Temporary Plant.

With $Q \in \mathcal{RH}_\infty$ a free parameter, we apply (4.74) employing F_∞ and for the moment *any* stabilizing H (as described above) to obtain all stabilizing controllers of $G_{tmp}(s)$:

$$K = LFT\left\{\left(\begin{array}{c|cc} \hat{A} + B_2F_\infty + H\hat{C}_2 & -H & -B_2 \\ \hline F_\infty & 0 & -I_{n_u} \\ -\hat{C}_2 & I_{n_y} & 0 \end{array}\right),\ Q\right\}. \tag{4.87}$$

The following description of all stable closed-loop systems for $G_{tmp}(s)$ follows from (4.75):

$$\Omega = LFT\left\{\left(\begin{array}{cc|cc} \hat{A} + B_2F_\infty & -B_2F_\infty & B_1 & B_2 \\ 0 & \hat{A} + H\hat{C}_2 & B_1 + HD_{21} & 0 \\ \hline C_1 - E_F + D_{12}F_\infty & -D_{12}F_\infty & 0 & D_{12} \\ 0 & \hat{C}_2 & D_{21} & 0 \end{array}\right),\ -Q\right\}, \tag{4.88}$$

where $\Omega = LFT\{G_{tmp},\ K\}$. It is easy to check that $(C_1 - E_F) + D_{12}F_\infty = 0$, from which it follows that we can remove the unobservable modes in (4.88) associated with the (stable) matrix $\hat{A} + B_2F_\infty = A_X$. Note therefore that

$$\Omega \;=\; LFT\left\{\left(\begin{array}{c|cc} \hat{A} + H\hat{C}_2 & B_1 + HD_{21} & 0 \\ \hline C_1 - E_F & 0 & D_{12} \\ \hat{C}_2 & D_{21} & 0 \end{array}\right),\ -Q\right\}. \tag{4.89}$$

$\mathcal{H}_\infty$ Synthesis in terms of the Youla Parameter.

Recall that we seek to describe all stable $Q(s)$ such that $\Omega(s) \in \mathcal{BH}_\infty$ and such that the interconnection (4.89) is internally stable. Therefore (4.89) actually constitutes an $\mathcal{H}_\infty$ controller synthesis problem, with the additional limitation imposed that the control law $Q(s)$ be stable. Note, however that it is still not a standard problem since D_{12} is nonstandard. However, we will next show that the nonstandard $\mathcal{H}_\infty$ synthesis problem associated with (4.89) can be solved by treating a rather simple standard $\mathcal{H}_\infty$ synthesis problem.

4.4 Construction of all Youla Parameters corresponding to $\mathcal{H}_\infty$ Controllers.

In this section, a parametrization is first derived of all Youla parameters $Q(s) \in \mathcal{RH}_\infty$ for the temporary plant $G_{tmp}(s)$ which ensure a closed-loop infinity norm bound on (4.89). Subsequently, that description is used to recover all $\mathcal{H}_\infty$ control laws for the original doubly nonstandard generalized plant.

It will be shown in Lemma 4.4.2 that a description of all solutions to the $\mathcal{H}_\infty$ synthesis problem defined by (4.89) can be found by first solving a *standard* output estimation $\mathcal{H}_\infty$ synthesis problem, which is defined in (4.109). Moreover, the existence of a solution to this problem is guaranteed by the existence of a solution to the original nonstandard problem. The result of solving the output estimation problem will be the description given in (4.140) of the *subset* of Youla parameters $Q(s)$ which solve the $\mathcal{H}_\infty$ problem given by (4.89) and which therefore correspond to $\mathcal{H}_\infty$ controllers for $G_{tmp}(s)$ (and by Lemma 4.3.3 also for $G(s)$). This description contains four degrees of freedom, a bounded real transfer function matrix $N(s)$ and three stable transfer function matrices $\hat{W}_i(s)$ $(i = 1, 2, 3)$.

Substitution of the above description of all parameters $Q(s)$ in the formula for all stabilizing controllers for $G_{tmp}(s)$, along with several simplifications, yields the final controller parametrization presented in Theorem 4.5.2 in section 4.5. This parametrization also has four degrees of freedom; the same bounded real transfer function matrix N, plus three *different* stable transfer function matrices W_i, which can be related to the free stable transfer function matrices matrices $\hat{W}_i$ $(i = 1, 2, 3)$ described above.

4.4.1 A Related Standard Output Estimation Problem.

We shall now show that all solutions to the (nonstandard) $\mathcal{H}_\infty$ problem associated with (4.89) may be described in terms of solutions to a related standard *output estimation* $\mathcal{H}_\infty$ problem. The following lemma is the key result, enabling the connection with the standard problem to be made. It should be noted that this lemma will later find application to the $(2,1)$ partition of the coefficient matrix in (4.89):

$$\left(\begin{array}{c|c} \hat{A} + H\hat{C}_2 & B_H \\ \hline \hat{C}_2 & D_{21} \end{array}\right). \tag{4.90}$$

Lemma 4.4.1 *Suppose one is given a realization of a generalized plant $G(s)$ as presented in (4.1) with D_{21} nonstandard. Let V_H be a matrix of full row rank whose row space spans the observable subspace of the pair $(D_{21}{}^{\perp}C_2, A - B_1 D_{21}{}^{\dagger}C_2)$.*

Consider the application of an arbitrary state-feedback transformation Ψ to the following realization

$$G_{21}(s) = C_2(sI - A)^{-1}B_1 + D_{21}, \tag{4.91}$$

and define the resulting transfer function matrix

$$\tilde{G}_{21}(s) = \tilde{C}_2(sI - \tilde{A})^{-1}B_1 + D_{21} \tag{4.92}$$

where $\tilde{A} = A + B_1\Psi$ and $\tilde{C}_2 = C_2 + D_{21}\Psi$.

Now consider subsequent application of an output injection transformation H to the above realization of $\tilde{G}_{21}$, resulting in a transfer function matrix

$$\hat{G}_{21}(s) = \tilde{C}_2 \left(sI - (\tilde{A} + H\tilde{C}_2)\right)^{-1} (B_1 + HD_{21}) + D_{21}. \tag{4.93}$$

Let H, without loss of generality, have the following form:

$$H = -B_1 D_{21}{}^{\dagger} + L_H D_{21}{}^{\perp} + E_H D_{21}{}^{\dagger}. \tag{4.94}$$

Suppose that for a given Ψ, L_H and E_H can be chosen such that both $V_H E_H = 0$ and the matrix $\tilde{A}_H = \tilde{A} + H\tilde{C}_2$ is stable.

Then with the definition $B_H = B_1 + HD_{21}$, the following identity holds:

$$\hat{G}_{21}(s) = \left(\begin{array}{c|c} \tilde{A} + H\tilde{C}_2 & B_H \\ \hline \tilde{C}_2 & D_{21} \end{array}\right) = D_{21} \left(\begin{array}{c|c} \tilde{A} + H\tilde{C}_2 & E_H \\ \hline D_{21}{}^{\dagger}\tilde{C}_2 & I_{n_w} \end{array}\right). \tag{4.95}$$

Proof: Observe that for any Ψ, the invariant zeros of the realizations of $G_{21}(s)$ and $\tilde{G}_{21}(s)$ as given in the lemma statement are identical. This can be argued as follows: note the equalities $\tilde{A} - B_1 D_{21}{}^{\dagger}\tilde{C}_2 = A - B_1 D_{21}{}^{\dagger} C_2$ and $D_{21}{}^{\perp}\tilde{C}_2 = D_{21}{}^{\perp} C_2$, from which it follows that the unobservable modes of $(A - B_1 D_{21}{}^{\dagger} C_2, D_{21}{}^{\perp} C_2)$ are identical to the unobservable modes of $(\tilde{A} - B_1 D_{21}{}^{\dagger}\tilde{C}_2, D_{21}{}^{\perp}\tilde{C}_2)$. It also follows that the pair $(\tilde{A} - B_1 D_{21}{}^{\dagger}\tilde{C}_2, D_{21}{}^{\perp}\tilde{C}_2)$ shares exactly the same observability canonical form as the pair $(A - B_1 D_{21}{}^{\dagger} C_2, D_{21}{}^{\perp} C_2)$, which was presented in (4.26) and (4.27).

With H chosen as described in the Lemma statement, the following identity holds:

$$\left(\begin{array}{c|c} \tilde{A} + H\tilde{C}_2 & B_H \\ \hline D_{21}{}^{\perp}\tilde{C}_2 & 0 \end{array}\right) = 0. \tag{4.96}$$

Some reasoning is now presented as to why this is the case. First note the following equality:

$$\tilde{A} + H\tilde{C}_2 = A - B_1 D_{21}{}^{\dagger} C_2 + L_H D_{21}{}^{\perp} C_2 + E_H D_{21}{}^{\dagger} C_2 + E_H \Psi. \tag{4.97}$$

We next express this equality in the basis associated with the observability canonical form introduced in (4.26) and (4.27). Recall the form of the similarity transformation associated with this canonical form:

$$U = \begin{pmatrix} U_H \\ V_H \end{pmatrix}. \tag{4.98}$$

Since we assume that $V_H E_H = 0$, it follows that there exists a matrix E_{H1} such that

$$E_H = U^{-1} \begin{pmatrix} E_{H1} \\ 0 \end{pmatrix}. \tag{4.99}$$

Recall from (4.27) that

$$D_{21}{}^{\perp} C_2 U^{-1} = \begin{pmatrix} 0 & \beta_H \end{pmatrix}, \tag{4.100}$$

and for convenience, we partition the matrix $D_{21}{}^{\dagger}C_2$ in this basis as follows:

$$D_{21}{}^{\dagger}C_2U^{-1} = \begin{pmatrix} \delta_1 & \delta_2 \end{pmatrix}. \tag{4.101}$$

Without loss of generality, L_H can be expressed as

$$L_H = U^{-1}\begin{pmatrix} L_{H1} \\ L_{H2} \end{pmatrix}, \tag{4.102}$$

and Ψ can be expressed as

$$\Psi = \begin{pmatrix} \psi_1 & \psi_2 \end{pmatrix} U. \tag{4.103}$$

From the above observations, it follows that (4.97) can be expressed in the basis associated with U as follows:

$$U(\tilde{A} + H\tilde{C}_2)U^{-1} = \begin{pmatrix} \alpha_0 + E_{H1}\psi_1 + E_{H1}\delta_1 & \alpha_{10} + L_{H1}\beta_H + E_{H1}\delta_2 + E_{H1}\psi_2 \\ 0 & \alpha_1 + L_{H2}\beta_H \end{pmatrix}.$$

From the above equation and the definition of U in (4.98), it follows that

$$V_H(\tilde{A} + H\tilde{C}_2) = (\alpha_1 + L_{H2}\beta_H)V_H. \tag{4.104}$$

Secondly, from (4.100) it follows that

$$D_{21}{}^{\perp}\tilde{C}_2 = \beta_H V_H. \tag{4.105}$$

Thirdly, observe that $B_1 + HD_{21} = E_H$, a fact which follows immediately from the structure of H given in (4.94). Since we assume that $V_H E_H = 0$, it follows that

$$V_H(B_1 + HD_{21}) = 0. \tag{4.106}$$

Consider the transpose of the realization on the left hand side of (4.96). We now apply Lemma 4.3.2 to $\hat{G}_{21}^T(s)$, identifying the following quantities: $V = V_H^T$, $A = \left(\tilde{A} + H\tilde{C}_2\right)^T$, $B = \left(D_{21}{}^{\perp}\tilde{C}_2\right)^T$ and $C = (B_1 + HD_{21})^T$. It can be seen that the conditions of that Lemma correspond to transposition of the identities (4.104), (4.105) and (4.106). The identity (4.96) follows immediately. from item **2** of Lemma 4.3.2.

Next note from equation (3.3) in section 3.1 of Chapter 3 that

$$D_{21}D_{21}{}^{\dagger} + (D_{21}{}^{\perp})^T D_{21}{}^{\perp} = I \tag{4.107}$$

and hence (trivially) that

$$\tilde{C}_2 = \left(D_{21}D_{21}{}^{\dagger} + (D_{21}{}^{\perp})^T D_{21}{}^{\perp}\right)\tilde{C}_2. \tag{4.108}$$

Making this substitution in the left hand side of (4.95) and noting (4.96) yields the right hand side of (4.95). □

A Temporary Assumption on the Structure of H.

We now seek to apply the above result to the $(2,1)$ partition of the coefficient matrix associated with the $\mathcal{H}_\infty$ control problem defined by (4.89), with the purpose of deriving a related standard problem. Recall from the discussion in section 4.3.3 that, provided an $\mathcal{H}_\infty$ controller exists for the original nonstandard generalized plant, we can be sure of the existence of at least one matrix H which stabilizes the matrix $\hat{A}+H\hat{C}_2$. However, it is not obvious in advance that one can find a stabilizing H which has the *structure* described in the above lemma and which thus enables the simplification given in (4.95). At this point in the argument, we *temporarily assume* that a stabilizing H exists with this structure. It will later be demonstrated that such an H does indeed exist and that this is a direct consequence of the assumed existence of a nonstandard controller for the original generalized plant.

Lemma 4.4.2 (The Related Output Estimation $\mathcal{H}_\infty$ Problem.)
With reference to Lemma 4.4.1, suppose that $\Psi = B_1^T X$ and that therefore $\tilde{A} = \hat{A}$ and $\tilde{C}_2 = \hat{C}_2$. Suppose also that H has the structure described in Lemma 4.4.1 and stabilizes $\hat{A}+H\hat{C}_2$.

Consider the $\mathcal{H}_\infty$ control problem defined by the linear fractional map

$$\hat{\Omega} = LFT\left\{\left(\begin{array}{c|cc} \hat{A}+H\hat{C}_2 & E_H & 0 \\ \hline C_1 - E_F & 0 & I_{n_z} \\ D_{21}{}^{\dagger}\hat{C}_2 & I_{n_w} & 0 \end{array}\right),\ \hat{K}(s)\right\}, \tag{4.109}$$

where stable controllers $\hat{K}(s) \in \mathcal{RH}_\infty$ are sought which ensure that both $\hat{\Omega}(s) \in \mathcal{BH}_\infty$ and that the connection is internally stable.

Then $Q(s) \in \mathcal{RH}_\infty$ is a solution to the $\mathcal{H}_\infty$ problem defined by (4.89) if and only if

$$Q(s) = -\begin{pmatrix} D_{12}{}^{\dagger} & D_{12}{}^{\perp} \end{pmatrix}\begin{pmatrix} \hat{K}(s) & \hat{W}_1 \\ \hat{W}_2 & \hat{W}_3 \end{pmatrix}\begin{pmatrix} D_{21}{}^{\dagger} \\ D_{21}{}^{\perp} \end{pmatrix}, \tag{4.110}$$

where $\hat{K}(s) \in \mathcal{RH}_\infty$ is a solution to the $\mathcal{H}_\infty$ problem defined by (4.109) and $\hat{W}_i(s) \in \mathcal{RH}_\infty$ (for $i = 1, 2, 3$).

Proof:
Necessity: Suppose one is given a transfer function matrix $Q(s) \in \mathcal{RH}_\infty$ which solves the $\mathcal{H}_\infty$ problem defined by (4.89). Note that with a choice of H as described in the lemma, one can apply (4.95) of Lemma 4.4.1 to the $(2,1)$ partition of the coefficient matrix of the linear fractional map in (4.89).

This allows the extraction of the feedthrough matrix D_{21} from the coefficient matrix in the linear fractional map into the feedback system part by right multiplying by D_{21} to give:

$$\Omega \ = \ LFT\left\{\left(\begin{array}{c|cc} \hat{A}+H\hat{C}_2 & B_1 + HD_{21} & 0 \\ \hline C_1 - E_F & 0 & D_{12} \\ D_{21}{}^{\dagger}\hat{C}_2 & I_{n_w} & 0 \end{array}\right),\ -QD_{21}\right\}. \tag{4.111}$$

In the coefficient matrix of the above linear fractional map, note that "$B_2'' = 0$. It follows that one can also extract the feedthrough matrix D_{12} from the coefficient matrix into to the feedback system part by left-multiplying to give $-D_{12}Q(s)D_{21}$. The result of this is a linear fractional map of the form (4.109). By identifying $\hat{K}(s) = -D_{12}Q(s)D_{21}$, we obtain $\hat{\Omega}(s) = \Omega(s)$, which by hypothesis is in $\mathcal{BH}_\infty$. Internal stability of the closed-loop is preserved after the extraction of the matrices D_{12} and D_{21} as described above. It follows therefore that $\hat{K}(s) = -D_{12}Q(s)D_{21} \in \mathcal{RH}_\infty$ is a solution to the $\mathcal{H}_\infty$ problem defined by (4.109).

Recall that parts **1** and **2** of Lemma 3.1.1 in Chapter 3 can be applied to D_{21} and D_{12}, respectively. The following identities result:

$$\begin{pmatrix} D_{12}^{\dagger} & D_{12}^{\perp} \end{pmatrix} \begin{pmatrix} D_{12} \\ (D_{12}^{\perp})^T \end{pmatrix} = I, \tag{4.112}$$

$$\begin{pmatrix} D_{21} & (D_{21}^{\perp})^T \end{pmatrix} \begin{pmatrix} D_{21}^{\dagger} \\ D_{21}^{\perp} \end{pmatrix} = I. \tag{4.113}$$

Note that it follows (trivially) from these identities that

$$\begin{aligned} Q(s) &= \begin{pmatrix} D_{12}^{\dagger} & D_{12}^{\perp} \end{pmatrix} \begin{pmatrix} D_{12} \\ (D_{12}^{\perp})^T \end{pmatrix} Q(s) \begin{pmatrix} D_{21} & (D_{21}^{\perp})^T \end{pmatrix} \begin{pmatrix} D_{21}^{\dagger} \\ D_{21}^{\perp} \end{pmatrix} \\ &= -\begin{pmatrix} D_{12}^{\dagger} & D_{12}^{\perp} \end{pmatrix} \begin{pmatrix} \hat{K}(s) & \hat{W}_1(s) \\ \hat{W}_2(s) & \hat{W}_3(s) \end{pmatrix} \begin{pmatrix} D_{21}^{\dagger} \\ D_{21}^{\perp} \end{pmatrix} \end{aligned} \tag{4.114}$$

where $\hat{W}_1(s) = -D_{12}Q(s)(D_{21}^{\perp})^T \in \mathcal{RH}_\infty$, $\hat{W}_2(s) = -(D_{12}^{\perp})^T Q(s) D_{21} \in \mathcal{RH}_\infty$ and $\hat{W}_3(s) = -(D_{12}^{\perp})^T Q(s)(D_{21}^{\perp})^T \in \mathcal{RH}_\infty$. This demonstrates that $Q(s)$ has the form described in (4.110).

Sufficiency: Suppose now that one is given a any $Q(s) \in \mathcal{RH}_\infty$ as described by (4.110) where $\hat{K}(s)$ is any stable solution of the $\mathcal{H}_\infty$ problem defined by (4.109) and $\hat{W}_i(s)$ (for $i = 1, 2, 3$) are stable free parameters. With this choice of $Q(s)$, an identical argument to that presented in the necessity proof above allows us to conclude that $\Omega(s) = \hat{\Omega}(s)$. Since $\hat{K}$ is an $\mathcal{H}_\infty$ controller for (4.109), it follows that $\Omega(s) = \hat{\Omega}(s) \in \mathcal{BH}_\infty$ and is internally stable. Observe that none of the $\hat{W}_i(s)$ appear in the closed-loop system $\Omega(s)$. We conclude that $Q(s)$ is a solution of the $\mathcal{H}_\infty$ problem defined by (4.89). □

4.4.2 Existence of Solutions to the Related Output Estimation Problem.

Recall the temporary assumption introduced in the previous subsection; that there exists a matrix H which has the special structure described in Lemma 4.4.1 and which simultaneously stabilizes the matrix $\tilde{A} + H\tilde{C}_2$. Subject to this assumption, we now exhibit conditions under which the $\mathcal{H}_\infty$ problem defined in (4.109) has solutions. We then show that if a solution to the original nonstandard $\mathcal{H}_\infty$ problem exists, then

- The temporary assumption on H is guaranteed to hold.

- A solution to the related output estimation problem defined in (4.109) is guaranteed to exist.

Necessary and Sufficient Conditions for the Existence of a Solution to the Related Output Estimation Problem.

In considering the coefficient matrix in (4.109) as a generalized plant of the type given in (4.1), we identify "D_{12}" with I_{n_z}, "A" with the (stable) matrix $\hat{A}+H\hat{C}_2$, "B_2" with the zero matrix and "D_{21}" with I_{n_w}. These special features allow more particularized $\mathcal{H}_\infty$ results to be obtained than the standard $\mathcal{H}_\infty$ results stated in Lemma 2.3.1 of Chapter 2. In fact, the task of synthesizing $\mathcal{H}_\infty$ control laws $\hat{K}(s)$ in (4.109) is a standard *output estimation* type problem (apart from the requirement that $\hat{K}(s)$ be stable). Controller existence and parametrization results for this special case of the output estimation problem are summarized in Lemma E.0.1 of Appendix E. In fact, $\mathcal{H}_\infty$ controllers in such cases are *always* stable, a fact which is essential in ensuring the stability of $\hat{K}(s)$ in (4.109).

By applying the existence conditions in item **1** of Lemma E.0.1 to (4.109), one obtains the algebraic Riccati equation:

$$\begin{aligned} 0 &= SA_{ZH}{}^T + A_{ZH}S \\ &+ S[(C_1 - E_F)^T(C_1 - E_F) - \hat{C}_2^T(D_{21}{}^\dagger)^T D_{21}{}^\dagger \hat{C}_2]S. \end{aligned} \tag{4.115}$$

Note that A_{ZH} appearing in this equation has the same form as the quantity defined in (4.12); $A_{ZH} = \hat{A}+H\hat{C}_2 - E_H D_{21}{}^\dagger \hat{C}_2 = A - B_1 D_{21}{}^\dagger C_2 + L_H D_{21}{}^\perp C_2$. Recall that A_{ZH} appears in the Riccati equation for Y

$$YA_{ZH}{}^T + A_{ZH}Y + Y[C_1{}^T C_1 - C_2{}^T(D_{21}{}^\dagger)^T D_{21}{}^\dagger C_2]Y = 0, \tag{4.116}$$

which in turn is part of the existence conditions given in Lemma 4.1.2.

To summarize:
Suppose $H = -B_1D_{21}{}^\dagger + L_H D_{21}{}^\perp + E_H D_{21}{}^\dagger$, with V_H and L_H chosen such that $V_H E_H = 0$ and $\hat{A}+H\hat{C}_2$ is stable. Then, the existence of a nonnegative definite stabilizing solution to (4.115) is necessary and sufficient for the existence of a solution to the output estimation $\mathcal{H}_\infty$ problem defined in (4.109).

Proof of the Existence of a Solution to the Related Problem.

We now show that the existence of $\mathcal{H}_\infty$ controllers for the original generalized plant (4.1) imply the existence of a nonnegative definite stabilizing solution of (4.115), and thereby the existence of an $\mathcal{H}_\infty$ controller for the generalized plant in (4.109). It will also become apparent that this solution enables the construction of a matrix H which stabilizes $\hat{A}+H\hat{C}_2$ and has the structure assumed earlier. It should be emphasized that the following lemma does *not* in any way depend on H having this structure, but only on the existence of a solution to the original nonstandard problem.

Lemma 4.4.3 *Suppose one is given a generalized plant $G(s)$ with realization (4.1) satisfying assumptions* **A.1**, **A.2** *and* **A.3** *and with both D_{12} and D_{21} nonstandard, i.e. $n_z < n_u$ and $n_w < n_y$.*

Let V_H be a matrix of full row rank whose row space spans the observable subspace of the pair $(D_{21}{}^{\perp}C_2, A - B_1 D_{21}{}^{\dagger}C_2)$ and let L_H be any matrix which stabilizes the observable subspace of this pair.

Suppose there exists an $\mathcal{H}_\infty$ controller for the nonstandard plant $G(s)$. Let X and Y denote the corresponding nonnegative definite stabilizing solutions of the algebraic Riccati equations (4.77) and (4.116), whose existence is guaranteed by Lemma 4.1.2.

Then it follows that:

1. *$S = (I - YX)^{-1}Y$ is a nonnegative definite stabilizing solution of (4.115).*
2. *$V_H S = 0$ and S is independent of L_H.*
3. *Let H_∞ be defined according to (4.94) in Lemma 4.4.1, with*

$$E_H = -S\hat{C}_2^T (D_{21}{}^{\dagger})^T, \tag{4.117}$$

and hence

$$H_\infty = -B_1 D_{21}{}^{\dagger} + L_H D_{21}{}^{\perp} - S\hat{C}_2^T (D_{21}{}^{\dagger})^T D_{21}{}^{\dagger}. \tag{4.118}$$

Then the matrix $\hat{A} + H_\infty \hat{C}_2$ is stable.

Proof:

1. Consider the Hamiltonian matrix associated with (4.115):

$$H_S = \begin{pmatrix} A_{ZH}{}^T & \left[(C_1 - E_F)^T (C_1 - E_F) - \hat{C}_2^T (D_{21}{}^{\dagger})^T D_{21}{}^{\dagger}\hat{C}_2\right] \\ 0 & -A_{ZH} \end{pmatrix}. \tag{4.119}$$

Recall from Lemma 4.1.2 that there exist nonnegative definite stabilizing solutions X and Y of the algebraic Riccati equations (4.77) and (4.116).

Application of the following similarity transformation to H_S

$$H_{\tilde{Y}} \;=\; \begin{pmatrix} I & X \\ 0 & I \end{pmatrix} H_S \begin{pmatrix} I & -X \\ 0 & I \end{pmatrix}, \tag{4.120}$$

results in another Hamiltonian matrix $H_{\tilde{Y}}$. By calculating the $n \times n$ block elements of this matrix, it can be easily checked that all but the $(2,2)$ block of $H_{\tilde{Y}}$ are the same as those of H_S. Straightforward algebra can be used to find the following formula for the $(2,2)$ block of $H_{\tilde{Y}}$:

$$C_1{}^T C_1 - C_2{}^T (D_{21}{}^{\dagger})^T D_{21}{}^{\dagger} C_2 - X L_H D_{21}{}^{\perp} C_2 - C_2{}^T (D_{21}{}^{\perp})^T L_H^T X - \mathcal{R}(X) \tag{4.121}$$

where

$$\mathcal{R}(X) = (A - B_2 D_{12}{}^{\dagger} C_1)^T X + +X(A - B_2 D_{12}{}^{\dagger} C_1) + X(B_1 B_1^T - B_2 D_{12}{}^{\dagger}(D_{12}{}^{\dagger})^T B_2^T)X. \tag{4.122}$$

Since X is a solution of the ARE (4.77) and since $XB_2D_{12}{}^{\perp} = 0$, it follows that $\mathcal{R}(X) = 0$.
Thus, the Hamiltonian matrix $H_{\tilde{Y}}$ corresponds with the following algebraic Riccati equation:

$$\begin{aligned}0 &= \tilde{Y}A_{ZH}{}^T + A_{ZH}\tilde{Y} + \tilde{Y}[C_1{}^T C_1 - C_2{}^T(D_{21}{}^{\dagger})^T D_{21}{}^{\dagger}C_2]\tilde{Y} \\ &\quad + \tilde{Y}[-XL_H D_{21}{}^{\perp}C_2 - C_2{}^T(D_{21}{}^{\perp})^T L_H^T X]\tilde{Y}. \end{aligned} \tag{4.123}$$

With Y the nonnegative definite stabilizing solution of the algebraic Riccati equation (4.116), we now show that $\tilde{Y} = Y$ is a stabilizing solution of (4.123). Recall from (4.19) in Lemma 4.1.2 that Y has the property $V_H Y = 0$, where V_H is is any matrix of full row rank whose row space spans the observable subspace of the pair $(D_{21}{}^{\perp}C_2, A - B_1D_{21}{}^{\dagger}C_2)$. In (4.27) it is stated that $D_{21}{}^{\perp}C_2 = \beta_H V_H$. From the last two sentences, one can conclude that $D_{21}{}^{\perp}C_2 Y = 0$. It can then be checked that $\tilde{Y} = Y$ is indeed a solution of (4.123). By hypothesis, $Y \geq 0$. The fact that Y is a stabilizing solution of (4.123) is inherited directly from the fact that it is a stabilizing solution of (4.116). It follows then that there exists a stable matrix $\Lambda_+ \in \mathbb{R}^{n \times n}$ such that

$$H_{\tilde{Y}}\begin{pmatrix} I \\ Y \end{pmatrix} = \begin{pmatrix} I \\ Y \end{pmatrix}\Lambda_+. \tag{4.124}$$

By substituting for $H_{\tilde{Y}}$ in this equation using the identity (4.120), one obtains the following identity after some simple rearrangement:

$$H_S\begin{pmatrix} I - XY \\ Y \end{pmatrix} = \begin{pmatrix} I - XY \\ Y \end{pmatrix}\Lambda_+. \tag{4.125}$$

Since $\rho(XY) < 1$ by hypothesis, it follows that the quantity $I - XY$ is invertible. It is a standard result that $S = Y(I - XY)^{-1}$ is then a solution of (4.115). That S is a *stabilizing* solution follows from (4.125) and the fact that Λ_+ is stable. Since both X and Y are nonnegative definite, S is also well defined and nonnegative definite. The standard identity $S = Y(I - XY)^{-1} = (I - YX)^{-1}Y$ can be used to show $S^T = S$ and confirms that S is as stated in item *1* of the Lemma.

2. It was demonstrated in Lemma 4.1.2 that $V_H Y = 0$. Recall from the proof of *1* that one valid formula for S is $S = Y(I - XY)^{-1}$. The stated result follows.

3. Observe the following identity

$$\hat{A} + H_\infty \hat{C}_2 = A_{ZH} + E_H D_{21}{}^{\dagger}\hat{C}_2. \tag{4.126}$$

This follows from the identity (4.97) in the proof of Lemma 4.4.1, with the choice $\Psi = B_1^T X$.

With E_H as chosen in the lemma statement, note that the ARE for S in (4.115) can be rewritten as follows:

$$0 = S(A_{ZH} + E_H D_{21}{}^{\dagger}\hat{C}_2)^T + (A_{ZH} + E_H D_{21}{}^{\dagger}\hat{C}_2)S$$

$$+S[(C_1 - E_F)^T(C_1 - E_F)]S + E_H E_H^T. \tag{4.127}$$

Recall that since S is a stabilizing solution of the ARE (4.115), the following matrix is stable:

$$A_{ZH} + E_H D_{21}{}^\dagger \hat{C}_2 + S[(C_1 - E_F)^T(C_1 - E_F)]. \tag{4.128}$$

It follows from the above fact that the following pair must be stabilizable:

$$\left(A_{ZH} + E_H D_{21}{}^\dagger \hat{C}_2, S(C_1 - E_F)^T\right). \tag{4.129}$$

With this in mind, along with the fact that $S \geq 0$, standard Lyapunov stability results can be applied to (4.127) to show that $A_{ZH} + E_H D_{21}{}^\dagger \hat{C}_2 = \hat{A} + H_\infty \hat{C}_2$ is also stable. □

Removal of the Temporary Assumption on the Structure of H.

Items *2* and *3* of the above lemma together present a choice of stabilizing output injection matrix $H = H_\infty$ in the Youla parametrization of stabilizing controllers for $G_{tmp}(s)$ which has the structure specified in Lemma 4.4.1. Thus we can now apply Lemma 4.4.2 with this choice of H. In summary, it is possible to treat the $\mathcal{H}_\infty$ control problem defined via (4.89) by treating that defined via (4.109).

Item *1* of the above lemma confirms the existence of a solution $\hat{K}(s)$ of the $\mathcal{H}_\infty$ control problem defined by (4.109). Recall that stability of $\hat{K}(s)$ is guaranteed by item 6 of Lemma E.0.1 in Appendix E. In the next section, we investigate the state-space description of all controllers $\hat{K}(s)$. This, together with the result of Lemma 4.4.2, enables all Youla parameters corresponding to $\mathcal{H}_\infty$ controllers of the temporary generalized plant to be found.

4.4.3 All Solutions to the Doubly Nonstandard $\mathcal{H}_\infty$ Problem.

In order to parametrize all solutions $\hat{K}(s)$ of the standard output estimation problem in (4.109), one can apply the controller formulae in Lemma E.0.1 of Appendix E. With reference to that lemma, we identify "L" in (E.5) with $-E_H - S\hat{C}_2^T(D_{21}{}^\dagger)^T$ which is identically zero owing to (4.117). It follows that one has the general form $\hat{K}(s) = LFT\{M, N\}$, where $N \in \mathcal{BH}_\infty$ and

$$M = \left(\begin{array}{c|cc} \hat{A} + H_\infty \hat{C}_2 & 0 & S(C_1 - E_F)^T \\ \hline -(C_1 - E_F) & 0 & I_{n_z} \\ -D_{21}{}^\dagger \hat{C}_2 & I_{n_w} & 0 \end{array}\right).$$

From this it follows that

$$\hat{K}(s) = M_{12}N(I - M_{22}N)^{-1} \tag{4.130}$$

where

$$M_{12}(s) = I - (C_1 - E_F)\Phi(s), \tag{4.131}$$

$$\begin{aligned} M_{22}(s) &= -D_{21}{}^{\dagger}\hat{C}_2\Phi(s), & (4.132)\\ \Phi(s) &= (sI - \hat{A} - H_\infty \hat{C}_2)^{-1} S(C_1 - E_F)^T \ \in \mathcal{RH}_\infty. & (4.133)\end{aligned}$$

From (4.110) in Lemma 4.4.2, it is evident that one can express each of the transfer function matrices $Q(s) \in \mathcal{RH}_\infty$ which solve the $\mathcal{H}_\infty$ problem in (4.89) in the following manner:

$$Q(s) = -\begin{pmatrix} D_{12}{}^{\dagger} & D_{12}{}^{\perp}\end{pmatrix} P(s) \begin{pmatrix} D_{21}{}^{\dagger} \\ D_{21}{}^{\perp}\end{pmatrix} \tag{4.134}$$

where

$$P(s) = \begin{pmatrix} M_{12}N(I - M_{22}N)^{-1} & \hat{W}_1 \\ \hat{W}_2 & \hat{W}_3 \end{pmatrix}, \tag{4.135}$$

with $\hat{W}_1, \hat{W}_2, \hat{W}_3 \in \mathcal{RH}_\infty$.

Observe next that $P(s)$ can be expressed in linear fractional form as:

$$P = \begin{pmatrix} M_{12} & 0 \\ L_F\Phi(s) & I \end{pmatrix} \begin{pmatrix} N & W_1 \\ W_2 & W_3 \end{pmatrix} \left\{ I - \begin{pmatrix} M_{22} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} N & W_1 \\ W_2 & W_3 \end{pmatrix} \right\}^{-1}, \tag{4.136}$$

where $W_i(s) \in \mathcal{RH}_\infty$ are free parameters which are related to $\hat{W}_i(s) \in \mathcal{RH}_\infty$ via the following equalities:

$$\begin{aligned} \hat{W}_1 &= M_{12}\left(N(I - M_{22}N)^{-1}M_{22} + I\right) W_1, & (4.137)\\ \hat{W}_2 &= (L_F\Phi(s)N + W_2)(I - M_{22}N)^{-1}, & (4.138)\\ \hat{W}_3 &= (L_F\Phi(s)N + W_2)(I - M_{22}N)^{-1}M_{22}W_1 + L_F\Phi W_1 + W_3. & (4.139)\end{aligned}$$

We now show that $\hat{W}_i \in \mathcal{RH}_\infty$ if and only if $W_i \in \mathcal{RH}_\infty$ $(i = 1, 2, 3)$ and thus that there is no loss of generality in the new description (4.136) of $P(s)$:
Note firstly from Lemma E.0.1 that both M_{12} and $(I - M_{22}N)^{-1}$ are unimodular matrices and in addition that each $M_{ij}(s) \in \mathcal{RH}_\infty$. Suppose now that $W_i \in \mathcal{RH}_\infty$ $(i = 1, 2, 3)$. It then follows directly from (4.137)–(4.139) that $\hat{W}_i \in \mathcal{RH}_\infty$.
Suppose now that $\hat{W}_i \in \mathcal{RH}_\infty$. It can be fairly easily shown from (4.137) that one can solve for $W_1 = (I - NM_{22})M_{12}^{-1}\hat{W}_1$, from which it follows that $W_1 \in \mathcal{RH}_\infty$. From (4.138), it follows that $W_2 = \hat{W}_2(I - M_{22}N) - L_F\Phi(s)N$ and therefore that $W_2 \in \mathcal{RH}_\infty$. From (4.139), it follows that $W_3 = \hat{W}_3 - \hat{W}_2 M_{22} W_1 - L_F\Phi(s)W_1$ and the stability of this matrix follows from the results established thus far.

Observe that $-D_{12}{}^{\dagger}M_{12} - D_{12}{}^{\perp}L_F\Phi = -D_{12}{}^{\dagger} - F_\infty\Phi$, which together with (4.134) allows one to deduce that

$$\begin{aligned} Q &= \begin{pmatrix} -D_{12}{}^{\dagger} - F_\infty\Phi, & -D_{12}{}^{\perp} \end{pmatrix} W \left\{ I - \begin{pmatrix} M_{22} & 0 \\ 0 & 0 \end{pmatrix} W \right\}^{-1} \begin{pmatrix} D_{21}{}^{\dagger} \\ D_{21}{}^{\perp} \end{pmatrix} \\ &= LFT\{M_Q, W\} & (4.140)\end{aligned}$$

where

$$W = \begin{pmatrix} N & W_1 \\ W_2 & W_3 \end{pmatrix} \tag{4.141}$$

and

$$M_Q = \left(\begin{array}{c|c:cc} \hat{A}+H\hat{C}_2 & 0 & S(C_1-E_F)^T & 0 \\ \hline -F_\infty & 0 & -D_{12}{}^\dagger & -D_{12}{}^\perp \\ \hdashline -D_{21}{}^\dagger \hat{C}_2 & D_{21}{}^\dagger & 0 & 0 \\ 0 & D_{21}{}^\perp & 0 & 0 \end{array}\right). \tag{4.142}$$

If we combine the description of all Youla parameters (4.140) with the formula for all $\mathcal{H}_\infty$ controllers for G_{tmp} given in (4.87), we obtain $K(s) = LFT\{M_K, W\}$ (corresponding to the description given in Theorem 4.5.2) where

$$\begin{aligned} M_K &= \left(\begin{array}{cc|ccc} \hat{A}+B_2F_\infty+H_\infty\hat{C}_2 & B_2F_\infty & -H_\infty & B_2D_{12}{}^\dagger & B_2D_{12}{}^\perp \\ 0 & \hat{A}+H_\infty\hat{C}_2 & 0 & S(C_1-E_F)^T & 0 \\ \hline F_\infty & F_\infty & 0 & D_{12}{}^\dagger & D_{12}{}^\perp \\ -D_{21}{}^\dagger\hat{C}_2 & -D_{21}{}^\dagger\hat{C}_2 & D_{21}{}^\dagger & 0 & 0 \\ -D_{21}{}^\perp\hat{C}_2 & 0 & D_{21}{}^\perp & 0 & 0 \end{array}\right) \\ &= \left(\begin{array}{cc|ccc} \hat{A}+B_2F_\infty+H_\infty\hat{C}_2 & 0 & -H_\infty & Z\hat{B}_2D_{12}{}^\dagger & B_2D_{12}{}^\perp \\ 0 & \hat{A}+H_\infty\hat{C}_2 & 0 & S(C_1-E_F)^T & 0 \\ \hline F_\infty & 0 & 0 & D_{12}{}^\dagger & D_{12}{}^\perp \\ -D_{21}{}^\dagger\hat{C}_2 & 0 & D_{21}{}^\dagger & 0 & 0 \\ -D_{21}{}^\perp\hat{C}_2 & D_{21}{}^\perp\hat{C}_2 & D_{21}{}^\perp & 0 & 0 \end{array}\right) \\ &= \left(\begin{array}{c|ccc} \hat{A}+B_2F_\infty+H_\infty\hat{C}_2 & -H_\infty & Z\hat{B}_2D_{12}{}^\dagger & B_2D_{12}{}^\perp \\ \hline F_\infty & 0 & D_{12}{}^\dagger & D_{12}{}^\perp \\ -D_{21}{}^\dagger\hat{C}_2 & D_{21}{}^\dagger & 0 & 0 \\ -D_{21}{}^\perp\hat{C}_2 & D_{21}{}^\perp & 0 & 0 \end{array}\right). \end{aligned}$$

The first expression for $M_K(s)$ is obtained using the star product formula in Appendix F. The second expression can then be obtained by applying a simple basis transformation to the first.

Now consider how the third expression can be obtained from the second. Recall from Lemma 4.4.3 that $V_H S = 0$. Identical reasoning to that which led to the identity (4.96) in the proof of Lemma 4.4.1 can be used to show the following equality:

$$D_{21}{}^\perp \hat{C}_2 \Phi(s) = 0. \tag{4.143}$$

From (4.143), it follows that the $(5,2)$ partition of the second expression for M_K (the matrix $D_{21}{}^\perp\hat{C}_2$) can be eliminated. The final expression for M_K can then be obtained by removing the stable unobservable modes corresponding to $\hat{A}+H_\infty\hat{C}_2$.

In the above we have $Z = (I-YX)^{-1}$, $S = ZY$ and $\hat{B}_2 = YC_1^T D_{12} + B_2$. Note here that H_∞ (defined according to Lemma 4.4.3) can be rewritten as

$$\begin{aligned} H_\infty &= -B_1D_{21}{}^\dagger - S\hat{C}_2^T(D_{21}{}^\dagger)^T D_{21}{}^\dagger + L_H D_{21}{}^\perp && (4.144) \\ &= Z\left(-B_1D_{21}{}^\dagger - YC_2^T(D_{21}{}^\dagger)^T D_{21}{}^\dagger\right) + L_H D_{21}{}^\perp. && (4.145) \end{aligned}$$

4.5 Summary of the Main Results.

For reference, we now summarize the main results concerning $\mathcal{H}_\infty$ controller existence and parametrization for nonstandard plants of cases 1, 2 and 3. These results should be compared with the nonstandard $\mathcal{H}_\infty$ results obtained using the augmented plant approach which are summarized in section 3.6 of Chapter 3. Except for some minor differences, the results are the same. A comparison of the results obtained via the two approaches is presented in the next section.

For convenience, we now restate the state-space $\mathcal{H}_\infty$ synthesis problem without signal dimension restrictions. Given a generalized plant $G(s)$, realized as follows:

$$G(s) = \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array} \right), \tag{4.146}$$

which satisfies the following assumptions:

A.1 (A, B_2) is stabilizable and (C_2, A) is detectable.

A.2 Both D_{12} and D_{21} are full rank.

A.3 Neither $G_{12}(s)$ nor $G_{21}(s)$, as described by the state space realization in (4.146), have imaginary axis invariant zeros.

Find all FDLTI control laws $K(s)$ which ensure that the closed loop mapping $T_{zw}^K(s)$ is internally stable and that $\|T_{zw}^K\|_\infty < 1$.

Riccati Equations for $\mathcal{H}_\infty$ Controller Existence and Parametrization:

- When D_{12} **is standard** (full column rank), consider

$$\begin{aligned} 0 = {} & X(A - B_2 {D_{12}}^\dagger C_1) + (A - B_2 {D_{12}}^\dagger C_1)^T X \\ & + X(B_1 B_1^T - B_2 {D_{12}}^\dagger ({D_{12}}^\dagger)^T B_2^T) X + C_1^T ({D_{12}}^\perp)^T {D_{12}}^\perp C_1, \end{aligned} \tag{4.147}$$

where ${D_{12}}^\dagger$ and ${D_{12}}^\perp$ are defined according to the identity

$$\begin{pmatrix} {D_{12}}^\dagger \\ {D_{12}}^\perp \end{pmatrix} \begin{pmatrix} D_{12} & ({D_{12}}^\perp)^T \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{4.148}$$

- When D_{12} **is nonstandard** (full row rank and $n_z < n_u$), consider

$$X A_{ZF} + {A_{ZF}}^T X + X(B_1 {B_1}^T - B_2 {D_{12}}^\dagger ({D_{12}}^\dagger)^T {B_2}^T) X = 0, \tag{4.149}$$

where ${D_{12}}^\dagger$ and ${D_{12}}^\perp$ are defined according to the identity

$$\begin{pmatrix} D_{12} \\ ({D_{12}}^\perp)^T \end{pmatrix} \begin{pmatrix} {D_{12}}^\dagger & {D_{12}}^\perp \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \tag{4.150}$$

and where

$$A_{ZF} \;=\; A - B_2{D_{12}}^{\dagger}C_1 + B_2{D_{12}}^{\perp}L_F \tag{4.151}$$

with L_F any matrix which stabilizes the controllable subspace of

$$(A - B_2{D_{12}}^{\dagger}C_1, B_2{D_{12}}^{\perp}). \tag{4.152}$$

- When $\boldsymbol{D_{21}}$ **is standard** (full row rank), consider

$$\begin{aligned} 0 = {} & Y(A - B_1{D_{21}}^{\dagger}C_2)^T + (A - B_1{D_{21}}^{\dagger}C_2)Y \\ & + Y(C_1^TC_1 - C_2^T({D_{21}}^{\dagger})^T{D_{21}}^{\dagger}C_2)Y + B_1{D_{21}}^{\perp}({D_{21}}^{\perp})^TB_1^T \end{aligned} \tag{4.153}$$

where ${D_{21}}^{\dagger}$ and ${D_{21}}^{\perp}$ are defined according to the identity

$$\begin{pmatrix} D_{21} \\ ({D_{21}}^{\perp})^T \end{pmatrix} \begin{pmatrix} {D_{21}}^{\dagger} & {D_{21}}^{\perp} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{4.154}$$

- When $\boldsymbol{D_{21}}$ **is nonstandard** (full column rank and $n_y > n_w$), consider

$$Y{A_{ZH}}^T + A_{ZH}Y + Y[{C_1}^TC_1 - {C_2}^T({D_{21}}^{\dagger})^T{D_{21}}^{\dagger}C_2]Y \;=\; 0, \tag{4.155}$$

where ${D_{21}}^{\dagger}$ and ${D_{21}}^{\perp}$ be defined according to the identity

$$\begin{pmatrix} {D_{21}}^{\dagger} \\ {D_{21}}^{\perp} \end{pmatrix} \begin{pmatrix} D_{21} & ({D_{21}}^{\perp})^T \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \tag{4.156}$$

and where

$$A_{ZH} \;=\; A - B_1{D_{21}}^{\dagger}C_2 + L_H{D_{21}}^{\perp}C_2 \tag{4.157}$$

with L_H any matrix which stabilizes the observable subspace of

$$({D_{21}}^{\perp}C_2, A - B_1{D_{21}}^{\dagger}C_2). \tag{4.158}$$

Nonstandard $\mathcal{H}_\infty$ Controller Existence and Parametrization Results.

Theorem 4.5.1 (Case 1) *Given a singly nonstandard generalized plant (4.146) which satisfies assumptions* **A.1, A.2** *and* **A.3** *for which D_{12} is nonstandard and D_{21} is standard, an $\mathcal{H}_\infty$ controller exists if and only if (4.149) and (4.153) have stabilizing solutions $X \geq 0$ and $Y \geq 0$ which satisfy $\rho(XY) < 1$.*

When it exists, the set of all $\mathcal{H}_\infty$ controllers can be expressed as:

$$K(s) = LFT\left\{ \left(\begin{array}{c|c:cc} \hat{A} + B_2F_\infty + H_\infty\hat{C}_2 & -H_\infty & Z\hat{B}_2{D_{12}}^{\dagger} & B_2{D_{12}}^{\perp} \\ \hline F_\infty & 0 & {D_{12}}^{\dagger} & {D_{12}}^{\perp} \\ \hdashline -E_{21}^{-\frac{1}{2}}\hat{C}_2 & E_{21}^{-\frac{1}{2}} & 0 & 0 \end{array} \right), \begin{pmatrix} N \\ W \end{pmatrix} \right\} \tag{4.159}$$

with free parameters

$$N(s) \in \mathcal{BH}_\infty^{n_z \times n_y}\ ,\ W(s) \in \mathcal{RH}_\infty^{(n_u - n_z)\times n_y}$$

and the definitions: $E_{21} = D_{21}D_{21}^T$,

$$\begin{array}{rclrcl} F_\infty &=& -D_{12}{}^\dagger C_1 + D_{12}{}^\perp L_F - D_{12}{}^\dagger (D_{12}{}^\dagger)^T B_2{}^T X, & \hat{A} &=& A + B_1 B_1^T X, \\ H_\infty &=& Z\left(-B_1 D_{21}{}^\dagger - Y C_2^T (D_{21}{}^\dagger)^T D_{21}{}^\dagger\right), & \hat{C}_2 &=& C_2 + D_{21} B_1^T X, \\ Z &=& (I - YX)^{-1}, & \hat{B}_2 &=& B_2 + Y C_1^T D_{12}. \end{array}$$

Proof: A proof is not presented here, however this result can be proven using identical reasoning to that employed in the proof of the case 2 result which has been presented. □

Theorem 4.5.2 (Case 2) *Given a generalized plant (4.146) which satisfies assumptions* **A.1**, **A.2** *and* **A.3** *and for which both D_{12} and D_{21} are nonstandard, an $\mathcal{H}_\infty$ controller exists if and only if (4.149) and (4.155) have stabilizing solutions $X \geq 0$ and $Y \geq 0$ which satisfy $\rho(XY) < 1$.*

When it exists, the set of all $\mathcal{H}_\infty$ controllers $K(s)$ can be expressed as:

$$K(s) = \qquad (4.160)$$

$$LFT\left\{\left(\begin{array}{c|c:cc} \hat{A} + B_2 F_\infty + H_\infty \hat{C}_2 & -H_\infty & Z\hat{B}_2 D_{12}{}^\dagger & B_2 D_{12}{}^\perp \\ \hline F_\infty & 0 & D_{12}{}^\dagger & D_{12}{}^\perp \\ \hdashline -D_{21}{}^\dagger \hat{C}_2 & D_{21}{}^\dagger & 0 & 0 \\ -D_{21}{}^\perp C_2 & D_{21}{}^\perp & 0 & 0 \end{array}\right), \begin{pmatrix} N & W_1 \\ W_2 & W_3 \end{pmatrix}\right\}$$

with free parameters

$$N(s) \in \mathcal{BH}_\infty^{n_z \times n_w},\ W_1(s) \in \mathcal{RH}_\infty^{n_z \times (n_y - n_w)},$$

$$W_2(s) \in \mathcal{RH}_\infty^{(n_u - n_z)\times n_w},\ W_3(s) \in \mathcal{RH}_\infty^{(n_u - n_z)\times(n_y - n_w)},$$

and the definitions:

$$\begin{array}{rclrcl} F_\infty &=& -D_{12}{}^\dagger C_1 + D_{12}{}^\perp L_F - D_{12}{}^\dagger (D_{12}{}^\dagger)^T B_2{}^T X, & \hat{A} &=& A + B_1 B_1^T X, \\ H_\infty &=& Z\left(-B_1 D_{21}{}^\dagger - Y C_2^T (D_{21}{}^\dagger)^T D_{21}{}^\dagger\right) + L_H D_{21}{}^\perp, & \hat{C}_2 &=& C_2 + D_{21} B_1^T X, \\ Z &=& (I - YX)^{-1}, & \hat{B}_2 &=& B_2 + Y C_1^T D_{12}. \end{array}$$

Proof: A proof of this result is contained in sections 4.3 and 4.4. □

Theorem 4.5.3 (Case 3) *Given a generalized plant (4.146) which satisfies assumptions* **A.1**, **A.2** *and* **A.3** *for which D_{12} is standard and D_{21} is nonstandard, an $\mathcal{H}_\infty$ controller exists if and only if (4.147) and (4.155) have stabilizing solutions $X \geq 0$ and $Y \geq 0$ which satisfy $\rho(XY) < 1$.*

When it exists, the set of all $\mathcal{H}_\infty$ controllers $K(s)$ can be expressed as:

$$K(s) = LFT\left\{\left(\begin{array}{c|c:c} \tilde{A} + \hat{B}_2 F_\infty + H_\infty C_2 & H_\infty & -\hat{B}_2 E_{12}^{-\frac{1}{2}} \\ \hline -F_\infty & 0 & E_{12}^{-\frac{1}{2}} \\ \hdashline D_{21}{}^{\dagger}\, \hat{C}_2\, Z & D_{21}{}^{\dagger} & 0 \\ D_{21}{}^{\perp} C_2 & D_{21}{}^{\perp} & 0 \end{array}\right), \begin{pmatrix} N & W \end{pmatrix}\right\} \tag{4.161}$$

with free parameters

$$N(s) \in \mathcal{BH}_\infty^{n_u \times n_w}, \quad W(s) \in \mathcal{RH}_\infty^{n_u \times (n_y - n_w)}$$

and the definitions: $E_{12} = D_{12}^T D_{12}$,

$$\begin{aligned} F_\infty &= (-D_{12}{}^{\dagger} C_1 - D_{12}{}^{\dagger}(D_{12}{}^{\dagger})^T B_2^T X)Z, & \tilde{A} &= A + Y C_1^T C_1, \\ H_\infty &= -B_1 D_{21}{}^{\dagger} + L_H D_{21}{}^{\perp} - Y C_2^T (D_{21}{}^{\dagger})^T D_{21}{}^{\dagger}, & \hat{C}_2 &= C_2 + D_{21} B_1^T X, \\ Z &= (I - YX)^{-1}, & \hat{B}_2 &= B_2 + Y C_1^T D_{12}. \end{aligned}$$

Proof: This result follows by applying the results for case 1 in Theorem 4.5.1 to the transpose of the case 3 plant. The transpose of the resulting controller parametrization furnishes the above description of all controllers for the case 3 plant. □

4.6 Interpretation and comparison of the Nonstandard $\mathcal{H}_\infty$ results.

In each of Chapters 3 and 4, a solution (existence conditions and a full controller parametrization) of the same nonstandard $\mathcal{H}_\infty$ control problem is presented. The derivations differ markedly in approach and in the type of techniques used. However, the solutions obtained bear a close relationship with each other. One objective of this section is to reconcile some minor differences in the solutions obtained in each chapter, thus demonstrating that they are in fact equivalent.

4.6.1 Canonical Forms and the Riccati Equations.

Calculation of the canonical forms displaying the invariant zeros of the subsystems $G_{12}(s)$ and $G_{21}(s)$ play an important role in both approaches. In both solutions of the nonstandard $\mathcal{H}_\infty$ problem, this canonical form is used to calculate the auxiliary matrices $\bar{C}_1/L_F$ and/or $\bar{B}_1/L_H$ which appear in the controller existence conditions and parametrization.

In the augmented plant approach, the augmentation matrix $\bar{C}_1$ must be chosen such such that it stabilizes the controllable subspace of the pair $(A - B_2 D_{12}{}^{\dagger} C_1, -B_2 D_{12}{}^{\perp})$. Similarly, in the Youla parametrization approach, a preliminary step is the calculation of a matrix L_F which stabilizes the controllable subspace of $(A - B_2 D_{12}{}^{\dagger} C_1, B_2 D_{12}{}^{\perp})$. Both $\bar{C}_1$ and L_F can be constructed with the aid of a controllability canonical form for the pair $(A - B_2{D_{12}}^{\dagger}C_1, B_2{D_{12}}^{\perp})$ which results from a similarity transformation T:

$$T^{-1}(A - B_2{D_{12}}^{\dagger}C_1)T = \begin{pmatrix} A_0 & 0 \\ A_{01} & A_1 \end{pmatrix}, \tag{4.162}$$

$$T^{-1}B_2{D_{12}}^{\perp} = \begin{pmatrix} 0 \\ \beta_F \end{pmatrix}. \tag{4.163}$$

Importantly, the invariant zeros of the given realization of $G_{12}(s)$ are displayed in this canonical form in that they are the eigenvalues of A_0. Recall that we choose in Chapter 3, we chose

$$\bar{C}_1 = -\begin{pmatrix} L_1 & L_2 \end{pmatrix} T^{-1}, \tag{4.164}$$

where L_2 is any matrix such that $A_1 + \beta_F L_2$ is stable and L_1 is arbitrary. Recall also that L_F, as chosen in Chapter 4 has exactly the same type of construction:

$$L_F = (L_{F1} \;\; L_{F2})T^{-1}, \tag{4.165}$$

where L_{F2} is chosen such that $\alpha_F = A_1 + \beta_F L_{F2}$ is stable and with L_{F1} arbitrary.

In summary, the class of all valid $\bar{C}_1$ is exactly the class of all $-L_F$ with L_F chosen as described earlier. Likewise, it follows using similar reasoning to the above, that the class of all valid $\bar{B}_1$ is the class of all $-L_H$.

Next recall the algebraic Riccati equations obtained via each approach when D_{12} is nonstandard:

$$XA_{ZF} + {A_{ZF}}^T X + X(\gamma^{-2}B_1{B_1}^T - B_2{D_{12}}^{\dagger}({D_{12}}^{\dagger})^T {B_2}^T)X = 0, \tag{4.166}$$

$$X_0 A_{ZX} + {A_{ZX}}^T X_0 + X_0(\gamma^{-2}B_1 B_1^T - B_2{D_{12}}^{\dagger}({D_{12}}^{\dagger})^T B_2^T)X_0 = 0, \tag{4.167}$$

with the definitions

$$A_{ZF} = A - B_2{D_{12}}^{\dagger}C_1 + B_2{D_{12}}^{\perp}L_F, \tag{4.168}$$

$$A_{ZX} = A - B_2{D_{12}}^{\dagger}C_1 - B_2{D_{12}}^{\perp}\bar{C}_1. \tag{4.169}$$

Since the set of possible L_F equals the set of $-\bar{C}_1$, the resulting set of A_{ZF} equals the set of all possible A_{ZX}. An analogous identification can be made between the possible sets of matrices A_{ZH} and A_{ZY}. Recall that when nonnegative definite stabilizing solutions of either of the nonstandard AREs exist, they are independent of the particular numerical values of $\bar{C}_1/L_F$ and/or $\bar{B}_1/L_H$, provided these matrices have the structure specified in the previous section. It follows immediately that the algebraic Riccati equations resulting from each approach always share the same nonnegative definite stabilizing solutions. Noting that nonnegative definite solutions to the algebraic Riccati equations are unique when they exist, we identify $X = X_0$ and $Y = Y_0$. Note that the subscript '0' was used in Chapter 4 to emphasize that the solutions X_0 and Y_0 are actually limiting values of the matrices X_ϵ and Y_ϵ, which are associated with the ϵ-augmented plant.

4.6.2 Solving the Riccati Equations.

Suppose a singly nonstandard problem having D_{12} full row rank is transformed to a standard problem by defining a new input signal u' via the preliminary transformation $u(t) = D_{12}{}^{\dagger}u'(t)$. This restricts the possible control actions to a subspace defined by the rowspace of D_{12}. In effect, some freedom in the input signal is ignored in this approach. This has been done with a view to first solving the standard $\mathcal{H}_\infty$ problem associated with the new input signal $u'(t)$, having the generalized plant:

$$G'(s) = \left(\begin{array}{c|cc} A & B_1 & B_2 D_{12}{}^{\dagger} \\ \hline C_1 & 0 & I \\ C_2 & D_{21} & 0 \end{array}\right). \tag{4.170}$$

The existence of a solution to the new standard problem for $G'(s)$ is clearly *sufficient* for the existence of the original nonstandard problem. If an $\mathcal{H}_\infty$ control law $K'(s)$ can be found for $G'(s)$, then $K(s) = D_{12}{}^{\dagger}K'(s)$ will be an $\mathcal{H}_\infty$ controller for the original nonstandard plant $G(s)$.

(note that the reverse is *not* true).

When one applies the standard $\mathcal{H}_\infty$ results to the standard generalized plant $G'(s)$, one obtains the following ARE:

$$(A - B_2 D_{12}{}^{\dagger} C_1)^T X' + X'(A - B_2 D_{12}{}^{\dagger} C_1) + X'(\gamma^{-2} B_1 B_1^T - B_2 D_{12}{}^{\dagger}(D_{12}{}^{\dagger})^T B_2^T)X' = 0. \tag{4.171}$$

The existence of a nonnegative definite stabilizing solution X' of (4.171) is a necessary and sufficient condition for the existence of an $\mathcal{H}_\infty$ control law for $G'(s)$. Recall that a stabilizing solution of (4.171) is one for which $A - B_2 D_{12}{}^{\dagger} C_1 + (\gamma^{-2} B_1 B_1^T - B_2 D_{12}{}^{\dagger}(D_{12}{}^{\dagger})^T B_2^T)X'$ is a stable matrix.

The above Riccati equation should be contrasted with the nonstandard $\mathcal{H}_\infty$ ARE (4.166).

One property of the nonnegative definite stabilizing solution X of the nonstandard ARE (4.166) which was highlighted in both the present chapter and in Chapter 3 was that it satisfies the identity:

$$X B_2 D_{12}{}^{\perp} = 0. \tag{4.172}$$

It follows immediately from this identity that that $X' = X$ also solves the ARE (4.171). Note, however that stabilizing solutions of (4.166) are not in general stabilizing solutions of (4.171).

It may be that we cannot find a nonnegative definite stabilizing solution of (4.171), even though a stabilizing solution of the equation (4.166) exists.

It should be noted that a stabilizing (but not necessarily simultaneously nonnegative definite) solution of the algebraic Riccati equation *always* exists. When D_{12} (D_{21}) is nonstandard, the constant term in the Riccati equation for X (Y) is absent. With this in mind, consider, the Hamiltonian matrix associated with the ARE (4.166):

$$H_X = \begin{pmatrix} A_{ZF}{}^T & (\gamma^{-2}B_1B_1{}^T - B_2D_{12}{}^\dagger(D_{12}{}^\dagger)^TB_2{}^T) \\ 0 & -A_{ZF} \end{pmatrix}. \tag{4.173}$$

Note that the eigenvalues of this matrix are those of A_{ZF} plus their reflections in the imaginary axis. Recall that the only closed right half plane eigenvalues of A_{ZF} correspond to zeros of $G_{12}(s)$. Recall also that imaginary axis invariant zeros are precluded by hypothesis (assumption **A.3**). It follows therefore that the Hamiltonian matrix H_X has no imaginary axis eigenvalues. This implies the existence of a stabilizing solution of the ARE. In fact, a straightforward calculation of the stable invariant subspace of the associated Hamiltonian matrix enables the construction of the ARE solution. For a summary of these ideas, see subsection 6.2.1.

Recall that any stabilizing solution of the nonstandard Riccati equation (4.166) must have the following structure:

$$X = (T^T)^{-1}\begin{pmatrix} \Psi & 0 \\ 0 & 0 \end{pmatrix}T^{-1}, \tag{4.174}$$

where T is a similarity transformation associated with the controllability canonical form for $(A - B_2D_{12}{}^\dagger C_1, B_2D_{12}{}^\perp)$ as given in (4.162). With partitioning conformal to that in (4.174), we write:

$$T^{-1}B_2D_{12}{}^\dagger = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \tag{4.175}$$

and

$$T^{-1}B_1 = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}. \tag{4.176}$$

There exists a nonnegative definite stabilizing solution X of (4.166) if and only if there exists a nonnegative definite stabilizing solution of the following ARE:

$$\Psi A_0 + A_0^T\Psi + \Psi\left(\gamma^{-2}\theta_1(\theta_1)^T - \gamma_1\gamma_1^T\right)\Psi = 0, \tag{4.177}$$

where A_0 appears in the canonical form (4.162) and its eigenvalues correspond to the invariant zeros of $G_{12}(s)$.

The scalar γ should not be confused with the matrix γ_1 in (4.177) which is defined as follows:

An equation of the same form as (4.177) appears in [86] where a so-called one-block state-feedback $\mathcal{H}_\infty$ control problem is addressed with $D_{12} = I$. It should be noted that the constant term in this ARE is also zero. The comparatively simple structure of the ARE also leads to its being solvable by direct calculation of the solution of two associated Lyapunov equations. The solution of these Lyapunov equations can also be used to deduce the minimum achievable $\mathcal{H}_\infty$ norm for the state-space system. A similar idea is employed in [16] where the Lyapunov equation solutions lead to a method by which the optimal $\mathcal{H}_\infty$ disturbance attenuation can be directly calculated for an output feedback $\mathcal{H}_\infty$ problem.

We now recount the approach suggested in [86] for solving AREs of the form (4.177).

In subsection 3.1.3 of Chapter 3, a whole family of nonsingular state space basis transformations T were introduced, each of which gives rise to a controllability canonical form as described above. There is a subclass of this family which results in a partitioning of the matrix A_0 as follows:

$$A_0 = \begin{pmatrix} A_0^1 & A_0^2 \\ 0 & A_0^3 \end{pmatrix}, \tag{4.178}$$

where A_0^1 is stable and A_0^3 is antistable (see [86]). Recall that A_0 has no imaginary axis eigenvalues (since $G_{12}(s)$ has no imaginary axis invariant zeros) and therefore that A_0^1 and A_0^3 are also free of imaginary axis eigenvalues. With reference to equation (4.177) and [86], we now introduce the following partitioning of θ_1 and γ_1 which is conformal to that in (4.178):

$$\theta_1 = \begin{pmatrix} \theta_1^1 \\ \theta_1^2 \end{pmatrix}, \tag{4.179}$$

$$\gamma_1 = \begin{pmatrix} \gamma_1^1 \\ \gamma_1^2 \end{pmatrix}. \tag{4.180}$$

It is shown in [86] that stabilizing solutions of (4.177) always have the following structure

$$\Psi = \begin{pmatrix} 0 & 0 \\ 0 & (S_3 - \gamma^{-2}T_3)^{-1}) \end{pmatrix}, \tag{4.181}$$

where $S_3 \geq 0$ and $T_3 \geq 0$ are solutions of the two Lyapunov equations:

$$-A_0^3 S_3 - (A_0^3)^T S_3 + \gamma_1^2(\gamma_1^2)^T = 0, \tag{4.182}$$

$$-A_0^3 T_3 - (A_0^3)^T T_3 + \theta_1^2(\theta_1^2)^T = 0. \tag{4.183}$$

It was shown in section 4.2 of the current chapter that (A, B_2) stabilizable implies that (A_0, γ_1) is also stabilizable. It then follows that (A_0^3, γ_1^2) is controllable, since (A_0, γ_1) stabilizable by definition means that all unstable modes must be controllable.

In the context of the nonstandard $\mathcal{H}_\infty$ problem, these results reveal a particularly simple structure to the problem. In effect, solving the Riccati equation corresponds to the solution of two Lyapunov equations of an order equal to the number of right half plane invariant zeros of the plant $G_{12}(s)$. This highlights the important role played by right half plane zeros in controller synthesis. It can be seen that the only nonzero component of the Riccati equation solution is due to the right half plane invariant zeros of the subsystem $G_{12}(s)$ of the generalized plant.

It may be of advantage to employ this approach in solving the Riccati equations as they appear in the existence conditions for the nonstandard problem at hand. This approach has the advantage that the optimal attenuation can be calculated immediately as $\gamma_{\min} = \{\lambda_{\max}(T_3 S_3^{-1})\}^{\frac{1}{2}}$ (again see [86]).

4.6.3 Parametrization of Control Laws.

Despite the differences in the derivations, the structure of the two nonstandard controller parametrizations is the same. Upon comparison, however it is apparent that there is a minor difference in the state-space formulae for the coefficient matrices in the linear fractional transformations used in the controller parametrizations. In particular, comparison of the case 2 controller parametrization results in Theorem 3.6.2 and Theorem 4.5.2 reveals a difference in the matrix "H_∞" as it is defined in each theorem. On the one hand, in the parametrized augmentation approach one has

$$H_\infty = Z_0\left(-B_1 D_{21}{}^{\dagger} - Y_0 C_2^T (D_{21}{}^{\dagger})^T D_{21}{}^{\dagger} - \bar{B}_1 D_{21}{}^{\perp}\right), \tag{4.184}$$

and yet in the present chapter

$$H_\infty = Z\left(-B_1 D_{21}{}^{\dagger} - Y C_2^T (D_{21}{}^{\dagger})^T D_{21}{}^{\dagger}\right) + L_H D_{21}{}^{\perp}. \tag{4.185}$$

We now resolve this apparent discrepancy. Our strategy shall be to start with (4.184) and show that it may expressed without loss of generality in the form (4.185).

First recall that $\bar{B}_1$ in (4.184) is any matrix which stabilizes the observable subspace of the pair $(-D_{21}{}^{\perp}C_2, A - B_1 D_{21}{}^{\dagger}C_2)$ and that L_H in (4.185) must be a matrix which stabilizes the observable subspace of $(D_{21}{}^{\perp}C_2, A - B_1 D_{21}{}^{\dagger}C_2)$. The matrices $\bar{B}_1$ and L_H may be chosen using the observability canonical form for the pair $(D_{21}{}^{\perp}C_2, A - B_1 D_{21}{}^{\dagger}C_2)$. There exists a nonsingular matrix U such that

$$\begin{aligned} U(A - B_1 D_{21}{}^{\dagger}C_2)U^{-1} &= \begin{pmatrix} \alpha_0 & \alpha_{10} \\ 0 & \alpha_1 \end{pmatrix}, && (4.186) \\ D_{21}{}^{\perp}C_2 U^{-1} &= \begin{pmatrix} 0 & \beta_H \end{pmatrix} && (4.187) \end{aligned}$$

where the pair (β_H, α_1) is observable. Using this canonical form, we can choose $\bar{B}_1$ as follows: $\bar{B}_1 = -U^{-1}\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$ with M_1 arbitrary and M_2 any matrix such that $\alpha_1 + M_2\beta_H$ is stable. Similarly, we can choose L_H as follows: $L_H = U^{-1}\begin{pmatrix} L_{H1} \\ L_{H2} \end{pmatrix}$ with L_{H1} arbitrary and L_{H2} any matrix such that $\alpha_1 + L_{H2}\beta_H$ is stable.

We now show that, given any $\bar{B}_1$ as described above, we can make a choice of L_H (also with structure described above) such that H_∞ as given in (4.185) equals the expression in (4.184). First note that it is easy to check that if we *define* $L_H = -Z\bar{B}_1$, the right hand side of (4.185) equals that of (4.184). However, it is not obvious at this point that this choice of L_H has the structure described in the previous paragraph. The following argument is intended to establish that it does.

$$\begin{aligned} & U(A - B_1 D_{21}{}^{\dagger}C_2 + L_H D_{21}{}^{\perp}C_2)U^{-1} && (4.188) \\ = & \begin{pmatrix} \alpha_0 & \alpha_{10} \\ 0 & \alpha_1 \end{pmatrix} + U(I - YX + YX)ZU^{-1}\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}\begin{pmatrix} 0 & \beta_H \end{pmatrix} && (4.189) \\ = & \begin{pmatrix} \alpha_0 & \alpha_{10} + M_1\beta_H \\ 0 & \alpha_1 + M_2\beta_H \end{pmatrix} + UYU^TU^{-T}XZU^{-1}\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}\begin{pmatrix} 0 & \beta_H \end{pmatrix}. && (4.190) \end{aligned}$$

Using a similar argument to that which was used to establish (3.31) in Lemma 3.3.2, it can be shown that

$$UYU^T = \begin{pmatrix} \Theta & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.191}$$

for some matrix Θ which is nonzero in general. It follows from (4.191) and (4.190) that

$$U(A - B_1 D_{21}{}^{\dagger} C_2 + L_H D_{21}{}^{\perp} C_2)U^{-1} = \begin{pmatrix} \times & \times \\ 0 & \alpha_1 + M_2\beta_H \end{pmatrix}, \tag{4.192}$$

where the entries denoted $\times$ are irrelevant to the argument. Since M_2 has already been chosen to stabilize $\alpha_1 + M_2\beta_H$, the identity (4.192) confirms that $L_H = -Z\bar{B}_1$ indeed stabilizes the observable subspace of the pair $(D_{21}{}^{\perp}C_2, A - B_1 D_{21}{}^{\dagger} C_2)$.

Suppose now that one is given a choice of $L_H = U^{-1}\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$ where M_2 is any matrix such that $\alpha_1 + M_2\beta_H$ is stable. With the choice $\bar{B}_1 = -Z^{-1}L_H$, it is easy to verify that the right hand side of (4.184) equals that of (4.185). That $\bar{B}_1 = -Z^{-1}L_H$ stabilizes the observable subspace of the pair $(D_{21}{}^{\perp}C_2, A - B_1 D_{21}{}^{\dagger}C_2)$ follows via similar arguments to those outlined above which were used to show the reverse implication.

In summary then, despite the fact that the techniques used to prove the controller parametrization results are quite different, the state space formulae are actually completely equivalent since the coefficient matrices of the linear fractional maps in the controller parametrizations can be made to equal each other by ensuring that the matrices L_H and $\bar{B}_1$ are related by the equality $L_H = -Z\bar{B}_1$. Importantly, this relation does not compromise the key structural properties of L_H and $\bar{B}_1$.

4.6.4 Relevance to Multiple Objective Design.

As was commented in the introduction of this thesis, a knowledge of the freedom in the parametrization of control laws for a given generalized plant and synthesis objective is of importance if secondary synthesis objectives are to be achieved. One means of approaching a mixed objective controller synthesis problem is to first find a parametrization of control laws satisfying one objective and then use the freedom in this parametrization to achieve other synthesis objectives. It was also noted in the introductionthat controller synthesis problems arising at each step using the above strategy can lead to a generalized plant which has more measurements than disturbances and/or more controls than objective signals. The additional degrees of freedom available in nonstandard $\mathcal{H}_\infty$ control as compared with the standard problem are particularly important from this perspective.

The stable rational transfer function matrix parameters W_1, W_2 and W_3 which appear in the parametrization of all doubly nonstandard controllers do not affect the closed-loop transfer function $T_{zw}(s)$ and hence do not affect the $\mathcal{H}_\infty$ norm. This fact can be verified by referring to the proof of the controller parametrization result in the present chapter. First note from subsection 4.3.2 that, given any control law $K(s)$, one can write the resulting closed loop operator as:

$$T_{zw}(s) = LFT\{\Theta, LFT\{G_{tmp}, K\}\}, \tag{4.193}$$

where state space realizations of $\Theta(s)$ and $G_{tmp}(s)$ are given in Lemma 4.3.3. Next note that with the choice $H = H_\infty$ in Lemma 4.4.2, it follows that:

$$LFT\{G_{tmp}, K\} = LFT\left\{\left(\begin{array}{c|cc} \hat{A} + H_\infty \hat{C}_2 & E_H & 0 \\ \hline \hat{C}_1 - E_F & 0 & I_{n_z} \\ D_{21}{}^\dagger \hat{C}_2 & I_{n_w} & 0 \end{array}\right), \hat{K}(s)\right\}, \tag{4.194}$$

where $\hat{K}$ is a solution to the $\mathcal{H}_\infty$ problem defined by the linear fractional map (4.194). Construction of $\hat{K}(s)$ is discussed in subsections 4.4.2 and 4.4.3. Note in particular that $\hat{K}(s)$ is completely independent of $W_1(s)$, $W_2(s)$ and $W_3(s)$. This fact, together with the expression (4.193), reveals that T_{zw} is also independent of $W_1(s)$, $W_2(s)$ and $W_3(s)$. It is in this sense that these parameters are *redundant* from the point of view of achieving the $\mathcal{H}_\infty$ objective. This is not true if other synthesis objectives are considered, however.

Whilst the additional freedom in the control laws cannot influence the nominal closed loop associated with the $\mathcal{H}_\infty$ objective, these parameters *do* have an effect on the closed-loop state dynamics. For this reason it is important to take into account the additional freedom in synthesis problems which include additional performance criteria, since these will in general be affected by the additional free parameters. One such example was discussed in Chapter 1 where the minimization of the $\mathcal{H}_2$ performance objective on a second input/output pair of the generalized plant was considered. If one were to consider a generalized plant which has a second input/output signal pair (v, n), then variation of W_1, W_2 and W_3 would in general influence T_{vn}.

It has *not* been the objective of this work to investigate the means by which such parameters can be chosen, but to reveal their existence and emphasize their significance for multiple objective control design. The question remains open as to how the free parameters might be chosen to improve closed-loop performance criteria associated with additional signals.

4.6.5 Comparing the Plant Augmentation and Youla Parametrization Proof Techniques.

We now compare the two approaches to the nonstandard $\mathcal{H}_\infty$ control problem which have been presented in the last two chapters. The two approaches have two major similarities. Firstly, the observations concerning the zeros of the systems $G_{12}(s)$ and $G_{21}(s)$ are common to both approaches. Also, both approaches draw upon the standard $\mathcal{H}_\infty$ theory at some point.

Derivation of control law parametrizations.

Note that in both cases, a lossless transformation is involved in obtaining output feedback controllers. This is a key structural reconfiguration which is essential in obtaining the particularly simple final formula for output feedback $\mathcal{H}_\infty$ control. In the case of the parametrized augmentation approach, the lossless decomposition is carried out *implicitly* since standard controller formulae (which rely on such a decomposition) are applied to the (standard) ϵ-augmented plant. In the Youla parametrization based approach, an *explicit* lossless decomposition of the original (nonstandard) plant is carried out.

The author believes that each of these approaches has its own value, insights and potential for future application to other problems. The techniques based on the Youla parametrization have been recently applied in [74] to nonstandard $\mathcal{H}_2$ controller synthesis. Similar techniques to those used in the augmented plant approach have been employed in [50] to derive controllers for a class of $\mathcal{H}_\infty$ servo problems.

4.7 Nonstandard $\mathcal{H}_2$ Synthesis via the Youla Parametrization.

The techniques used in the first part of this chapter to obtain nonstandard $\mathcal{H}_\infty$ results can also be used to obtain controller parametrization results for nonstandard $\mathcal{H}_2$ problems. For reference, we now summarize these results without proof (see e.g. [74] for a proof of the doubly nonstandard result).

State Space $\mathcal{H}_2$ Synthesis without Signal Dimension Restrictions.

Given a generalized plant $G(s)$, realized as follows:

$$G(s) = \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right), \tag{4.195}$$

which satisfies the following assumptions:

A.1 (A, B_2) is stabilizable and (C_2, A) is detectable.

A.2 Both D_{12} and D_{21} are full rank.

A.3 Neither $G_{12}(s)$ nor $G_{21}(s)$, as described by the state space realization in (4.195), have imaginary axis invariant zeros.

Find all FDLTI control laws $K(s)$ which ensure that the closed loop mapping $T_{zw}^K(s)$ is internally stable and that $\|T_{zw}^K\|_2$ is minimized.

Riccati Equations for $\mathcal{H}_2$ Controller Existence and Parametrization:

- When D_{12} **is standard** (full column rank), consider

$$\begin{aligned} 0 = {} & X(A - B_2 D_{12}{}^{\dagger} C_1) + (A - B_2 D_{12}{}^{\dagger} C_1)^T X \\ & - X B_2 D_{12}{}^{\dagger} (D_{12}{}^{\dagger})^T B_2^T X + C_1^T (D_{12}{}^{\perp})^T D_{12}{}^{\perp} C_1, \end{aligned} \tag{4.196}$$

  where $D_{12}{}^{\dagger}$ and $D_{12}{}^{\perp}$ are defined according to the identity

$$\begin{pmatrix} D_{12}{}^{\dagger} \\ D_{12}{}^{\perp} \end{pmatrix} \begin{pmatrix} D_{12} & (D_{12}{}^{\perp})^T \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{4.197}$$

- When D_{12} **is nonstandard** (full row rank and $n_z < n_u$), consider

$$X A_{ZF} + A_{ZF}{}^T X - X B_2 D_{12}{}^{\dagger} (D_{12}{}^{\dagger})^T B_2 X \;=\; 0, \tag{4.198}$$

  where $D_{12}{}^{\dagger}$ and $D_{12}{}^{\perp}$ are defined according to the identity

$$\begin{pmatrix} D_{12} \\ (D_{12}{}^{\perp})^T \end{pmatrix} \begin{pmatrix} D_{12}{}^{\dagger} & D_{12}{}^{\perp} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \tag{4.199}$$

  and where

$$A_{ZF} \;=\; A - B_2 D_{12}{}^{\dagger} C_1 + B_2 D_{12}{}^{\perp} L_F \tag{4.200}$$

  with L_F any matrix which stabilizes the controllable subspace of $(A - B_2 D_{12}{}^{\dagger} C_1, B_2 D_{12}{}^{\perp})$.

- When D_{21} **is standard** (full row rank), consider

$$\begin{aligned} 0 = {} & Y(A - B_1 D_{21}{}^{\dagger} C_2)^T + (A - B_1 D_{21}{}^{\dagger} C_2) Y \\ & - Y C_2^T (D_{21}{}^{\dagger})^T D_{21}{}^{\dagger} C_2 Y + B_1 D_{21}{}^{\perp} (D_{21}{}^{\perp})^T B_1^T \end{aligned} \tag{4.201}$$

  where $D_{21}{}^{\dagger}$ and $D_{21}{}^{\perp}$ are defined according to the identity

$$\begin{pmatrix} D_{21} \\ (D_{21}{}^{\perp})^T \end{pmatrix} \begin{pmatrix} D_{21}{}^{\dagger} & D_{21}{}^{\perp} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{4.202}$$

- When D_{21} **is nonstandard** (full column rank and $n_y > n_w$), consider

$$Y A_{ZH}{}^T + A_{ZH} Y - Y C_2^T (D_{21}{}^{\dagger})^T D_{21}{}^{\dagger} C_2 Y \;=\; 0, \tag{4.203}$$

  where $D_{21}{}^{\dagger}$ and $D_{21}{}^{\perp}$ be defined according to the identity

$$\begin{pmatrix} D_{21}{}^{\dagger} \\ D_{21}{}^{\perp} \end{pmatrix} \begin{pmatrix} D_{21} & (D_{21}{}^{\perp})^T \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \tag{4.204}$$

  and where

$$A_{ZH} \;=\; A - B_1 D_{21}{}^{\dagger} C_2 + L_H D_{21}{}^{\perp} C_2 \tag{4.205}$$

with L_H any matrix which stabilizes the observable subspace of $(D_{21}{}^{\perp}C_2, A - B_1D_{21}{}^{\dagger}C_2)$.

Nonstandard $\mathcal{H}_\infty$ Controller Existence and Parametrization Results.

Theorem 4.7.1 (Case 1) *Given a generalized plant (4.195) which satisfies assumptions* **A.1, A.2** *and* **A.3** *for which D_{12} is nonstandard and D_{21} is standard, the algebraic Riccati equations (4.198) and (4.201) have stabilizing solutions $X \geq 0$ and $Y \geq 0$ and an optimal $\mathcal{H}_2$ controller exists.*

The set of all optimal $\mathcal{H}_2$ controllers $K(s)$ can be expressed as:

$$K(s) = LFT\left\{\left(\begin{array}{c|c:cc} A+B_2F_2+H_2C_2 & -H_2 & B_2D_{12}{}^{\dagger} & B_2D_{12}{}^{\perp} \\ \hline F_2 & 0 & D_{12}{}^{\dagger} & D_{12}{}^{\perp} \\ \hdashline -E_{21}^{-\frac{1}{2}}C_2 & E_{21}^{-\frac{1}{2}} & 0 & 0 \end{array}\right), \begin{pmatrix} 0 \\ W \end{pmatrix}\right\} \tag{4.206}$$

with free parameter

$$W(s) \in \mathcal{RH}_\infty^{(n_u-n_z)\times n_y}$$

and the definitions: $E_{21} = D_{21}D_{21}^T$,

$$\begin{aligned} F_2 &= -D_{12}{}^{\dagger}C_1 + D_{12}{}^{\perp}L_F - D_{12}{}^{\dagger}(D_{12}{}^{\dagger})^T B_2{}^T X, && (4.207)\\ H_2 &= -B_1D_{21}{}^{\dagger} - YC_2^T(D_{21}{}^{\dagger})^T D_{21}{}^{\dagger}. && (4.208)\end{aligned}$$

Theorem 4.7.2 (Case 2) *Given a generalized plant (4.195) which satisfies assumptions* **A.1, A.2** *and* **A.3,** *for which both D_{12} and D_{21} are nonstandard, the algebraic Riccati equations (4.198) and (4.203) have stabilizing solutions $X \geq 0$ and $Y \geq 0$ and an optimal $\mathcal{H}_2$ controller exists.*

The set of all optimal $\mathcal{H}_2$ controllers $K(s)$ can be expressed as:

$$K(s) = LFT\left\{\left(\begin{array}{c|c:cc} A+B_2F_2+H_2C_2 & -H_2 & B_2D_{12}{}^{\dagger} & B_2D_{12}{}^{\perp} \\ \hline F_2 & 0 & D_{12}{}^{\dagger} & D_{12}{}^{\perp} \\ \hdashline -D_{21}{}^{\dagger}C_2 & D_{21}{}^{\dagger} & 0 & 0 \\ -D_{21}{}^{\perp}C_2 & D_{21}{}^{\perp} & 0 & 0 \end{array}\right), \begin{pmatrix} 0 & W_1 \\ W_2 & W_3 \end{pmatrix}\right\} \tag{4.209}$$

with free parameters

$$W_1(s) \in \mathcal{RH}_\infty^{n_z\times(n_y-n_w)},\ W_2(s) \in \mathcal{RH}_\infty^{(n_u-n_z)\times n_w},\ W_3(s) \in \mathcal{RH}_\infty^{(n_u-n_z)\times(n_y-n_w)}$$

and the definitions:

$$\begin{aligned} F_2 &= -D_{12}{}^{\dagger}C_1 + D_{12}{}^{\perp}L_F - D_{12}{}^{\dagger}(D_{12}{}^{\dagger})^T B_2{}^T X, && (4.210)\\ H_2 &= -B_1D_{21}{}^{\dagger} - YC_2^T(D_{21}{}^{\dagger})^T D_{21}{}^{\dagger} + L_H D_{21}{}^{\perp}. && (4.211)\end{aligned}$$

Theorem 4.7.3 (Case 3) *Given a generalized plant (4.195) which satisfies assumptions* **A.1**, **A.2** *and* **A.3** *for which D_{12} is standard and D_{21} is nonstandard, the algebraic Riccati equations (4.196) and (4.203) have stabilizing solutions $X \geq 0$ and $Y \geq 0$ and an opitimal $\mathcal{H}_2$ controller exists.*

The set of all optimal $\mathcal{H}_2$ controllers $K(s)$ can be expressed as:

$$K(s) = LFT\left\{\left(\begin{array}{c|c:c} A + B_2F_2 + H_2C_2 & H_2 & -B_2E_{12}^{-\frac{1}{2}} \\ \hline -F_2 & 0 & E_{12}^{-\frac{1}{2}} \\ \hdashline {D_{21}}^{\dagger} C_2 & {D_{21}}^{\dagger} & 0 \\ {D_{21}}^{\perp} C_2 & {D_{21}}^{\perp} & 0 \end{array}\right), \begin{pmatrix} 0 & W \end{pmatrix}\right\} \quad (4.212)$$

with free parameter

$$W(s) \in \mathcal{RH}_{\infty}^{n_u \times (n_y - n_w)}$$

and the definitions: $E_{12} = D_{12}^T D_{12}$,

$$\begin{aligned} F_2 &= -{D_{12}}^{\dagger} C_1 - {D_{12}}^{\dagger} ({D_{12}}^{\dagger})^T B_2^T X, & (4.213) \\ H_2 &= -B_1 {D_{21}}^{\dagger} + L_H {D_{21}}^{\perp} - Y C_2^T ({D_{21}}^{\dagger})^T {D_{21}}^{\dagger}. & (4.214) \end{aligned}$$

Part III

Spectral Factorization Algorithms.

Chapter 5

The Boundary of the $\mathcal{H}_\infty$ Constraint in Multiple Objective Robust Control.

Summary.

The aim of the present chapter is to investigate one particular aspect of multiple objective controller synthesis problems in which an $\mathcal{H}_\infty$ bound is sought on one input/output signal pair of the generalized plant. We shall focus on the mixed $\mathcal{H}_2$ / $\mathcal{H}_\infty$ synthesis problem which was introduced in subsection 1.2.2 of Chapter 1. The main conclusion of the present chapter is that the closed loop resulting from many multiple objective robust control laws must necessarily lie on the boundary of the $\mathcal{H}_\infty$ constraint. In other words, the closed loop transfer function associated with the $\mathcal{H}_\infty$ objective is not only bounded real but also achieves the specified bound on its infinity norm. This leads to the study of state space spectral factorization of spectral matrices which have imaginary circle zeros which is pursued in the subsequent chapter.

5.1 The Mixed $\mathcal{H}_2$ / $\mathcal{H}_\infty$ Synthesis Objective.

We now consider the *dual-objective controller synthesis* problem depicted in Figure 5.1 which was introduced in subsection 1.2.2 of Chapter 1. Suppose one has constructed a generalized plant $G(s)$ consisting of a nominal plant model and weighting functions which have been chosen and configured to reflect disturbances, system uncertainty and closed loop design objectives. In constructing the generalized plant, two distinct (vector-valued) exogenous normalized disturbance signals have been considered which we label $w(t)$ and $n(t)$. Associated with these two signals are the objective signals $z(t)$ and $v(t)$, respectively. Here we consider the task of finding internally stabilizing control laws $K(s)$ which achieve a closed loop $\mathcal{H}_\infty$ criterion with respect to the signal pair $\{w(t), z(t)\}$, whilst *simultaneously* achieving another, possibly different, criterion with respect to the other input/output pair $\{n(t), v(t)\}$.

As before, we limit our search to control laws which are linear, time-invariant and

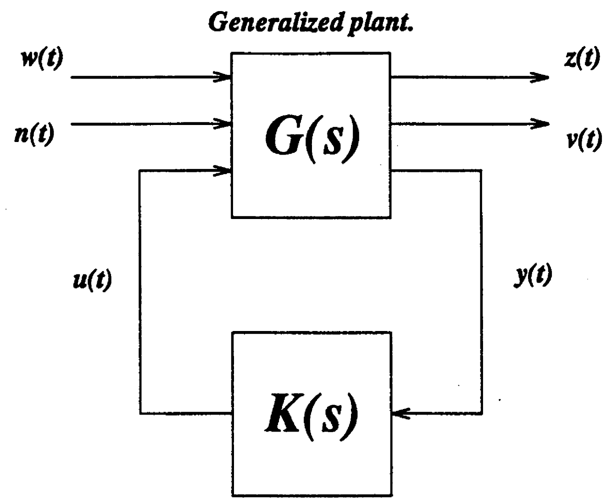

Figure 5.1: Multiple objective design framework with linear time invariant generalized plant and control law.

finite dimensional. Given any such control law $K(s)$, we call the resulting closed loop operator T^K, noting that

$$\begin{pmatrix} z(s) \\ v(s) \end{pmatrix} = T^K(s) \begin{pmatrix} w(s) \\ n(s) \end{pmatrix}. \tag{5.1}$$

Let this operator be partitioned according to the dimensions of the signals associated with the control objectives. Let $T^K_{zw}(s)$ and $T^K_{vn}(s)$ denote the $(1,1)$ and $(2,2)$ block-partitions of $T^K(s)$, having dimensions $n_z \times n_w$ and $n_v \times n_n$ respectively. By obtaining bounds on the infinity-norm of T^K_{zw}, the benefits of the linear robust $\mathcal{H}_\infty$ design methodology are sought. Since there is considerable freedom associated with $\mathcal{H}_\infty$ control laws, it is anticipated that there should be ample opportunity to use that freedom to achieve another performance objective on the operator $T^K_{vn}(s)$ in addition to achieving $\|T^K_{zw}\|_\infty \leq \gamma$. Minimization of the quantity $\|T^K_{vn}\|_2$ is adopted as the second performance objective.

Recall that $\mathcal{K}$ denotes the set of linear, time-invariant internally stabilizing control laws for the generalized plant $G(s)$. Let $\mathcal{K}^\gamma$ denote the set of γ-admissible $\mathcal{H}_\infty$ control laws:

$$\mathcal{K}^\gamma = \left\{ K | \; K \in \mathcal{K} \;\text{ and }\; \|T^K_{zw}\|_\infty < \gamma \right\}. \tag{5.2}$$

Next we consider the set $\tilde{\mathcal{K}}^\gamma \subset \mathcal{K}$ which denotes the set of all control laws which are either γ-admissible or *achieve* the closed-loop γ-bound with respect to the input/output signal pair $\{w(t), z(t)\}$;

$$\tilde{\mathcal{K}}^\gamma = \left\{ K | \; K \in \mathcal{K} \;\text{ and }\; \|T^K_{zw}\|_\infty \leq \gamma \right\}. \tag{5.3}$$

It will shortly be shown that it is necessary to consider this set, which is larger than $\mathcal{K}^\gamma$, when considering *optimal* closed loop $\mathcal{H}_2$ performance subject to a closed-loop $\mathcal{H}_\infty$

constraint.

$\mathcal{H}_\infty$ Controllers with Optimal Nominal $\mathcal{H}_2$ Performance.

Consider the following set of linear time invariant control laws:

$$\mathcal{K}^* = \left\{ K |\ K \in \mathcal{K},\ \ \|T_{zw}^K\|_\infty \leq \gamma, \text{and}\ \ \|T_{vn}^K\|_2\ \leq\ \|T_{vn}^{\hat{K}}\|_2\ \ \forall\ \hat{K} \in \tilde{\mathcal{K}}^\gamma \right\}. \tag{5.4}$$

In summary, these control laws provide the best possible nominal noise attenuation in an $\mathcal{H}_2$ sense with respect to the signal pair $\{n(t), v(t)\}$, whilst providing a guaranteed (non-strict) γ-bound on the $\mathcal{H}_\infty$ norm associated with the signal pair $\{w(t), z(t)\}$.

Relevance of the Mixed $\mathcal{H}_2$ / $\mathcal{H}_\infty$ Problem in Control Design.

In subsection 1.2.2 of Chapter 1, one rationale was given for solving the mixed $\mathcal{H}_2$ / $\mathcal{H}_\infty$ problem in terms of a design example. There is was shown that the generalized plant had disturbance signals which were of fundamentally differing nature. An $\mathcal{H}_\infty$ synthesis objective was used to quantify reference tracking performance, whilst an $\mathcal{H}_2$ synthesis objective was used to describe the effect of sensor noise on the system. We refer the reader to Chapter 1 for further discussion of this problem.

We now move on to introduce another control design scenario in which the mixed $\mathcal{H}_2$ / $\mathcal{H}_\infty$ objective would be of benefit. Suppose one has a nominal model which describes the dynamics of a process adequately for *most* of the time. However, every so often there are short-term deviations in the process dynamics from the nominal behaviour, whose magnitude and time of incidence cannot be predicted exactly in advance.

Firstly we would like to ensure good set-point tracking and disturbance attenuation during periods where the nominal model is accurate. For example, this may be important in ensuring good product quality during the normal operation of a process. Next suppose that a generalized plant has been constructed such that the objective of minimizing $\|T_{vn}^K\|_2$ adequately reflects the process performance objectives for the nominal model.

One could try applying $\mathcal{H}_2$ optimal control methods directly to the plant. However, it is well known (see [26]) that the closed-loop stability of $\mathcal{H}_2$ optimal control laws can be extremely sensitive to plant perturbations. Should the closed loop process dynamics become unstable during a period in which the nominal process model is inaccurate, and if this period is sufficiently long, then process variables may take on values which are either unsafe or which make recovery to normal operating conditions very slow, with a potential cost in terms of product quality. *We would therefore like to find a control law which, in addition to providing good nominal $\mathcal{H}_2$ performance, provides a guarantee of stability in the face of variations from nominal process behaviour.*

It should be noted that during a period of uncertain process behaviour, the process itself is time-varying and possibly nonlinear, whilst the nominal plant model, the generalized plant and the controller are all linear and time-invariant. An important feature of the $\mathcal{H}_\infty$ design methodology is that, in addition to providing a guaranteed stability margin against linear time invariant uncertainties, controllers can be designed which guarantee robust stability in the face of potentially *nonlinear* and *time-varying* modelling errors (see e.g. Chapter 3 of [34]). Suppose that a generalized plant has been constructed such that certain of its generalized disturbance and generalized objective signals can be associated with uncertain process behaviour. Denote this signal pair as $\{w(t), z(t)\}$. The intermittent variations in process behaviour are modelled as being due to a mapping $w(t) = X_n z(t)$, where X_n is an uncertain time-varying and/or nonlinear operator. If a bound is available on the *incremental gain* $\Gamma(X_n) < 1$ of the uncertainty, then one can design $\mathcal{H}_\infty$ control laws which provide robust stability (for a definition of incremental gain, see for example Chapter 3 of [34]). The *small gain theorem* says that a sufficient condition for robust stability in these circumstances is that $\Gamma(\Delta)\|T_{zw}^K\|_\infty < 1$.

The Mixed $\mathcal{H}_2$ / $\mathcal{H}_\infty$ Controller Synthesis Problem.

Suppose one is content to synthesize just *one* control law in $\mathcal{K}^*$, this being sufficient to satisfy the described objectives. Quite aside from the task of finding *all* controllers in $\mathcal{K}^*$, the problem of finding just one element is difficult. The difficulty of the mixed $\mathcal{H}_2$ / $\mathcal{H}_\infty$ controller synthesis problem is evidenced by the many attempts at solving it in the literature, and still the apparent lack of a straightforward and reliable design algorithm. Whilst it is now a long-standing problem, results are just emerging in the literature which point towards such algorithms (see e.g. [94] and [95]). The mixed $\mathcal{H}_2$ / $\mathcal{H}_\infty$ state-feedback controller synthesis problem was posed in [91] (see Problem A of that paper) where state feedback control laws were sought. In [91], it was observed that optimal mixed $\mathcal{H}_2$ / $\mathcal{H}_\infty$ state-feedback control laws may be *dynamic* and not static as is the case when the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ criteria are treated separately. This contrasts with the single objective $\mathcal{H}_2$ and $\mathcal{H}_\infty$ synthesis in which constant state-feedback laws are generally available (provided the state is measureable).

$\mathcal{H}_\infty$ Controllers which are Simultaneous Globally $\mathcal{H}_2$ Optimal.

There is one circumstance under which the synthesis of controllers in $\mathcal{K}^*$ is particularly simple; it is when the globally optimal $\mathcal{H}_2$ controller is also γ-admissible. Suppose we remove the constraint that $K \in \tilde{\mathcal{K}}^\gamma$ and design a *globally* optimal $\mathcal{H}_2$ control law K_2. Suppose also that, fortuitously, $\|T_{zw}^{K_2}\|_\infty \leq \gamma$. Clearly in this case $K \in \mathcal{K}^*$ and the mixed $\mathcal{H}_2$ / $\mathcal{H}_\infty$ problem is trivially solved. In fact, this case is the main subject of [91] (see Problem B in that paper), where conditions are developed in terms of state space plant data for the existence of a control law which is simultaneously globally $\mathcal{H}_2$ optimal and γ-admissible. The main difficulty with the mixed $\mathcal{H}_2$ / $\mathcal{H}_\infty$ problem is associated with those cases when the globally optimal $\mathcal{H}_2$ controller is *not* also γ-admissible.

A property of the $\mathcal{H}_\infty$ Norm of Mixed $\mathcal{H}_2$ / $\mathcal{H}_\infty$ controllers.

The controller synthesis problem for $\mathcal{K}^*$ is *not* addressed explicitly in this thesis. Instead, the approach taken here is to gain a better appreciation of a particular *property* of the set $\mathcal{K}^*$, and thus of the synthesis objective. The particular property investigated is the infinity norm of the closed loop system which results from optimal mixed $\mathcal{H}_2$ / $\mathcal{H}_\infty$ controllers.

Given that $K^* \in \mathcal{K}^* \quad \Rightarrow \quad \|T_{zw}^{K^*}\|_\infty \leq \gamma$, *when will it be true that* $\|T_{zw}^{K^*}\|_\infty = \gamma$ *?*

It is hoped that an answer to this question might help provide a basis for a mixed $\mathcal{H}_2$ / $\mathcal{H}_\infty$ synthesis algorithm or, in the very least, contribute to an understanding of why, despite many attempts in the literature, a simple and computationally straightforward solution to the mixed synthesis problem has not yet been found. An understanding of the $\mathcal{H}_\infty$ norm of mixed controllers should provide one reference point in the search for a controller synthesis algorithm. When implemented, mixed $\mathcal{H}_2$ / $\mathcal{H}_\infty$ control laws must create a closed loop system which achieves the infinity norm bound. In state-space, a result which can be used to check the $\mathcal{H}_\infty$ norm bound is the bounded real lemma, which says that satisfaction of the bound is equivalent to the existence of a strong solution of an algebraic Riccati equation. Solving the Riccati equation when the norm bound is achieved is numerically more difficult than is the case when the bound is strict. Most standard ARE solvers do not work in the latter case. However, algorithms are now available in the literature which are able to solve the ARE when the bound is achieved. In the two Chapters which follow this one, another such algorithm is developed which is iterative and has known convergence properties.

5.2 The Closed Loop Infinity Norm for Mixed $\mathcal{H}_2$ / $\mathcal{H}_\infty$ Controllers.

We shall see that elements of $\mathcal{K}^*$ can be characterized in a very specific way in terms of the infinity norm of the associated closed loop matrix T_{zw}^K: in general $\|T_{zw}^K\|_\infty = \gamma$. In Chapter 6, an algorithm is developed which is intended to enable this property to be checked using state-space calculations, given a state space description of the closed loop operator T_{zw}^K. Such a realization might, for example, result from a candidate optimal mixed controller $K(s)$ being connected to the generalized plant. This approach has been taken in the hope that the resulting algorithm will be of utility in the future development of algorithms for the synthesis of state space realizations of mixed $\mathcal{H}_2$ / $\mathcal{H}_\infty$ controllers.

Theorem 5.2.1 *Let $\mathcal{K}^*$ be the set of mixed $\mathcal{H}_2$ /$\mathcal{H}_\infty$ controllers as defined in (5.4) for a linear time invariant generalized plant $G(s)$ with input and output signals partitioned as in Figure 5.1. Suppose one is given any linear time invariant optimal mixed control law $K^* \in \mathcal{K}^*$, then if $K^*(s)$ is not simultaneously a globally optimal $\mathcal{H}_2$ controller, then K^* results in a closed loop which satisfies*

$$\|T_{zw}^{K^*}\|_\infty = \gamma. \tag{5.5}$$

Before attempting to prove this theorem, we establish two lemmas. The first concerns the effect of controller perturbations on the closed loop $\mathcal{H}_2$ norm and the second concerns the effect of such perturbations on the closed loop $\mathcal{H}_\infty$ norm. We shall use the well-known Youla parametrization of all stabilizing controllers to prove the following results. We now recount the main features of this parametrization which are relevant to the proofs.

Given a FDLTI generalized plant $G(s)$ with control input $u(t)$ and measured output $y(t)$, note the following facts:

- All control laws which internally stabilize $G_{22}(s)$ can be expressed as follows
$$K(s) = LFT\{M(s), Q(s)\}, \tag{5.6}$$
where $M(s)$ is a coefficient matrix which can be constructed from a *coprime factorization* of $G_{22}(s)$ and where $Q(s) \in \mathcal{RH}_\infty$ is a free parameter. (see e.g. Appendix A of [34] for an account of coprime factorization and the parametrization of all stabilizing controllers.)

- The above linear fractional map is an invertible mapping between control laws which internally stabilize $G_{22}(s)$ and stable parameters $Q(s)$.

- Suppose the generalized plant has disturbance signal $w(t)$ and objective signal $z(t)$, then when implemented, any internally stabilizing control law has a closed loop transfer function with respect to the signal pair $\{w(t), z(t)\}$ which can be written as
$$T^K_{zw}(s) = T^{zw}_1 + T^{zw}_2 Q T^{zw}_3 \tag{5.7}$$
where $T^{zw}_i(s)$ $(i = 1, 2, 3)$ are transfer function matrices which can be constructed from $G(s)$.

- If the generalized plant $G(s)$ contains only stable weighting functions, then the transfer function matrices $T^{zw}_i(s)$ $(i = 1, 2, 3)$ as defined above are stable.

Lemma 5.2.1 *Consider a linear time invariant generalized plant $G(s)$ with input and output signals partitioned as in Figure 5.1. Let $K(s) \in \mathcal{K}$ be any linear time invariant stabilizing control law which is not globally optimal in the $\mathcal{H}_2$ sense. Consider also the Youla parametrization of all stabilizing control laws for $G(s)$ and let $Q(s) \in \mathcal{RH}_\infty$ be the unique parameter associated with $K(s)$;*
$$K(s) = LFT\{M(s), Q(s)\}, \tag{5.8}$$
where $M(s)$ is a suitable coefficient matrix.
Then for any $\epsilon > 0$, there exists another Youla parameter $\tilde{Q}(s) \in \mathcal{RH}_\infty$ with $\|\tilde{Q} - Q\|_\infty <$

ϵ such that the associated linear time invariant stabilizing control law

$$\tilde{K} = LFT\{M(s), \tilde{Q}(s)\} \tag{5.9}$$

has better $\mathcal{H}_2$ performance than $K(s)$ in the sense that

$$\|T_{vn}^{\tilde{K}}\|_2 < \|T_{vn}^{K}\|_2. \tag{5.10}$$

Proof: First define the map $\mathcal{F}(\cdot) : \mathcal{RH}_\infty \to \mathbb{R}$ as

$$\begin{aligned} \mathcal{F}(Q) &= \|LFT\{G_{vn}, LFT\{M,Q\}\}\|_2 & (5.11)\\ &= \|T_1^{vn} + T_2^{vn} Q T_3^{vn}\|_2. & (5.12)\end{aligned}$$

The transfer function matrices T_i^{vn} $(i = 1, 2, 3)$ are those associated with all internally stable closed loops which map $n(t)$ to $v(t)$. Note that $\mathcal{F}(\cdot)$ has only been defined for arguments in $\mathcal{RH}_\infty$, since clearly we are only interested in internally stabilizing control laws.

Next note that the function $\mathcal{F}(\cdot)$ is *convex* in its argument; with $Q_1, Q_2 \in \mathcal{RH}_\infty$ and $\lambda \in [0, 1]$,

$$\begin{aligned} \mathcal{F}(\lambda Q_1 + (1-\lambda)Q_2) &= \|\lambda\,(T_1^{vn} + T_2^{vn} Q_1 T_3^{vn}) + (1-\lambda)\,(T_1^{vn} + T_2^{vn} Q_2 T_3^{vn})\|_2 \\ &\leq \lambda\mathcal{F}(Q_1) + (1-\lambda)\mathcal{F}(Q_2). & (5.13)\end{aligned}$$

Since $\|\cdot\|_2$ is a norm, the triangle inequality holds, which has been used to establish the above formula.

By hypothesis, the given control law K is not globally $\mathcal{H}_2$ optimal. In other words, Q does *not* globally minimize $\mathcal{F}(\cdot)$. Therefore there exists at least one $\hat{Q} \in \mathcal{RH}_\infty$ such that

$$\mathcal{F}(\hat{Q}) < \mathcal{F}(Q). \tag{5.14}$$

Recall, that it is assumed in the lemma statement that some number $\epsilon > 0$ has been given. Should the Youla parameter $\hat{Q}$ with the property (5.14) also satisfy $\|\hat{Q} - Q\|_\infty < \epsilon$, then with $\tilde{Q} = \hat{Q}$, the result stated in the lemma holds. Suppose instead that $\|\hat{Q} - Q\|_\infty \geq \epsilon$. It remains to show that there exists a $\tilde{Q}$ such that both $\|\tilde{Q} - Q\|_\infty < \epsilon$ and

$$\mathcal{F}(\tilde{Q}) < \mathcal{F}(Q). \tag{5.15}$$

Given $\hat{Q}$ and Q as defined above, define the family of Youla parameters $Q^\eta = \eta\hat{Q} + (1-\eta)Q$ where $\eta \in (0, 1]$. Observe the following inequalities

$$\begin{aligned} \mathcal{F}(Q^\eta) &\leq \eta\mathcal{F}(\hat{Q}) + (1-\eta)\mathcal{F}(Q) & (5.16)\\ &< \eta\mathcal{F}(Q) + (1-\eta)\mathcal{F}(Q) = \mathcal{F}(Q) & (5.17)\end{aligned}$$

The first inequality follows directly from the convexity of $\mathcal{F}(\cdot)$. The second inequality follows from (5.14). Thus each element of the family Q^η performs better in an $\mathcal{H}_2$ sense than Q.

Observe next that $Q^\eta - Q = \eta(\hat{Q} - Q)$. Since $\hat{Q}$ is not identically equal to Q and both are stable, it follows that $\|\hat{Q} - Q\|_\infty$ is finite and nonzero. By choosing any $\eta > 0$ as follows $\eta < \min\left\{\epsilon\,\|\hat{Q} - Q\|_\infty^{-1}, 1\right\}$, it follows that $\|Q^\eta - Q\|_\infty < \epsilon$. By then choosing $\tilde{Q} = Q^\eta$ the proof of the lemma is complete. □

Lemma 5.2.2 *Consider a linear time invariant generalized plant $G(s)$ with input and output signals partitioned as in Figure 5.1. Let $K(s) \in \mathcal{K}^\gamma$ be any γ-admissible control law. Consider also the Youla parametrization of all stabilizing control laws for $G(s)$ and let $Q(s) \in \mathcal{RH}_\infty$ be the unique parameter associated with $K(s)$:*

$$K(s) = LFT\{M(s), Q(s)\}, \tag{5.18}$$

where $M(s)$ is a suitable coefficient matrix.
Then there exists a constant $\eta > 0$ such that if $\hat{Q}(s) \in \mathcal{RH}_\infty$ with $\|\hat{Q} - Q\|_\infty < \eta$, it follows that

$$\hat{K} = LFT\{M(s), \hat{Q}(s)\} \;\in\; \mathcal{K}^\gamma. \tag{5.19}$$

Proof: Take any $\hat{Q}(s) \in \mathcal{RH}_\infty$ and define the associated stabilizing controller $\hat{K}$ as in (5.19). Consider the resulting closed loop operator associated with the $\mathcal{H}_\infty$ objective:

$$T_{zw}^{\hat{K}} \;=\; T_1^{zw} + T_2^{zw} Q T_3^{zw}. \tag{5.20}$$

By writing $T_{zw}^{\hat{K}} = T_{zw}^{K} + T_{zw}^{\hat{K}} - T_{zw}^{K}$, one can deduce from the triangle inequality for the norm $\|\cdot\|_\infty$ that

$$\|T_{zw}^{\hat{K}}\|_\infty \;\leq\; \|T_{zw}^{K}\|_\infty + \|T_{zw}^{\hat{K}} - T_{zw}^{K}\|_\infty. \tag{5.21}$$

Next note from (5.20) that

$$\begin{aligned} \|T_{zw}^{\hat{K}} - T_{zw}^{K}\|_\infty &= \|T_2^{zw}(\hat{Q} - Q)T_3^{zw}\|_\infty \\ &\leq \|T_2^{zw}\|_\infty \|\hat{Q} - Q\|_\infty \|T_3^{zw}\|_\infty. \end{aligned} \tag{5.22}$$

Note that since each of the transfer function matrices $T_i^{zw}(s)$ $(i = 1, 2, 3)$ are stable, their $\mathcal{H}_\infty$ norms are well defined. The same is true for $(\hat{Q} - Q)$. The inequality in (5.22) follows from the *submultiplicative* property of the norm $\|\cdot\|_\infty$; given any transfer function matrices $M(s)$ and $N(s)$ with finite infinity norm, then $\|MN\|_\infty \leq \|M\|_\infty \|N\|_\infty$.

Since, by hypothesis, $\|T_{zw}^{K}\|_\infty < \gamma$, it follows that there exists a real constant $0 < \epsilon^* \leq \gamma$ such that $\|T_{zw}^{K}\|_\infty = \gamma - \epsilon^*$. Note therefore from (5.21) and (5.22) that

$$\|T_{zw}^{\hat{K}}\|_\infty \;\leq\; \gamma - \epsilon^* + \|T_2^{zw}\|_\infty \|\hat{Q} - Q\|_\infty \|T_3^{zw}\|_\infty. \tag{5.23}$$

By choosing

$$\eta = \left(\|T_2^{zw}\|_\infty \|T_3^{zw}\|_\infty\right)^{-1} \epsilon^*, \tag{5.24}$$

it follows that $\|\hat{Q} - Q\|_\infty < \eta$ ensures that $\|T_{zw}^{\hat{K}}\|_\infty < \gamma$. This establishes the desired result. □

Proof of Theorem 5.2.1:
We now combine the results of Lemmas 5.2.1 and 5.2.2 to establish this result.

Let $K^*(s)$ be as described in the theorem statement and let Q^* denote the corresponding unique Youla parameter.

Suppose, contrary to the stated result, that $\|T_{zw}^{K^*}\|_\infty < \gamma$, i.e. $K^* \in \mathcal{K}^\gamma$.

Application of Lemma 5.2.2 to K^*, allows us to conclude that there exists a constant $\eta > 0$ such that if $\hat{Q} \in \mathcal{RH}_\infty$ and $\|\hat{Q} - Q^*\|_\infty < \eta$, then

$$\hat{K} = LFT\{M(s), \hat{Q}(s)\} \in \mathcal{K}^\gamma \subset \tilde{\mathcal{K}}^\gamma. \tag{5.25}$$

Recall that K^* is by hypothesis not globally optimal. It follows from Lemma 5.2.1 that one can find a control law arbitrarily close to K^* which has better $\mathcal{H}_2$ performance than K^*. For *arbitrary* $\epsilon > 0$, there exists a $\tilde{Q}$ with $\|\tilde{Q} - Q^*\|_\infty < \epsilon$ such that

$$\tilde{K} = LFT\{M(s), \tilde{Q}(s)\} \tag{5.26}$$

outperforms K^* in an $\mathcal{H}_2$ -norm sense.

Take any ϵ such that $0 < \epsilon < \eta$ and let $\tilde{Q}$ be defined according to Lemma 5.2.1; then the associated controller $\tilde{K}$ is in $\tilde{\mathcal{K}}^\gamma$ *and* has superior $\mathcal{H}_2$ performance to K^*. This is a contradiction since K^* has, by definition, optimal $\mathcal{H}_2$ performance within the set $\tilde{\mathcal{K}}^\gamma$ □

In summary, a characteristic of those mixed $\mathcal{H}_2$ /$\mathcal{H}_\infty$ control laws $K^* \in \mathcal{K}^*$ which are not simultaneously globally $\mathcal{H}_2$ optimal controllers, is that they achieve the closed loop infinity norm bound γ.

Theorem 5.2.1 has motivated the work in Chapter 6 on *spectral factorization with imaginary axis zeros.* In the remainder of this section we shall briefly review the connection between spectral factorization and the $\mathcal{H}_\infty$ constraint, a topic which is discussed in more detail in Chapter 6.

5.3 The Bounded Real Lemma and Spectral Matrices with Imaginary Axis Zeros.

In $\mathcal{H}_\infty$ control, a specified bound is sought on the infinity norm of selected closed loop transfer function matrices: one seeks to synthesize control laws subject to the *constraint* that the closed loop operator $T_{zw}^K(s)$ is γ-bounded real. Recall that a transfer function matrix $M(s)$ is called *γ-bounded real* (*strictly γ-bounded real*) if it is stable and $\|M(s)\|_\infty \leq \gamma$ ($\|M(s)\|_\infty < \gamma$). It has been pointed out in the previous section that closed loop systems which *achieve* the infinity norm bound arise when considering multiple objective robust control problems. *In essence, the closed loop system is on the* boundary *of the closed-loop $\mathcal{H}_\infty$ constraint set.* In such circumstances, the stable closed-loop operator $T_{zw}^K(s)$ satisfies $\bar{\sigma}(T_{zw}^K(j\omega)) = \gamma$, for at least one frequency $\omega \in [-\infty, \infty]$.

The Bounded Real Constraint expressed in terms of a Spectral Matrix.

It is well known that a *spectral matrix* can be associated with a γ-bounded real transfer function matrix $M(s)$:

$$\Phi(s) = \gamma^2 I - M^T(-s)M(s). \tag{5.27}$$

Note in particular that $\Phi(j\omega) \geq 0$ for all $\omega \in \mathbb{R} \cup \{\infty\}$ if and only if $\|M\|_\infty \leq \gamma$ (see [11] and [34]). Observe that if $M(j\omega)$ *achieves* the γ-bound at one or more points on the imaginary axis (including at infinity), then $\Phi(s)$ loses rank at these points. In other words, there exists a vector $\eta \neq 0$ such that $\Phi(j\omega)\eta = 0$ and $\Phi(s)$ has a transmission zero at $j\omega$.

State Space formulation of the Bounded Real Constraint.

Suppose one is given a linear time invariant control law K which internally stabilizes a generalized plant $G(s)$, giving rise to the closed loop system $T_{zw}^K(s)$. If one is given state-space realizations of $G(s)$ and $K(s)$, it is then easy to explicitly calculate a state space realization of the closed loop:

$$T_{zw}^K(s) = C(K)(sI - A(K))^{-1}B(K) + D(K), \tag{5.28}$$

where each of the matrices $A(K), B(K), C(K)$ and $D(K)$ are functions of the state-space data for $G(s)$ and $K(s)$ (see e.g. Chapter 4 of [34]). The very least we would like to be able to do at this point is to determine whether $M(s) = T_{zw}^K(s)$ is bounded real.

We now review how the *spectral property* $\Phi(j\omega) \geq 0$ can be interpreted in state-space terms, given a state-space realization of a bounded real transfer function matrix $M(s) = D + C(sI - A)^{-1}B$. The aim is to develop an algorithm which is capable of efficiently checking whether $M(s)$ is γ-bounded real. Let $\Phi(s)$ be defined as in (5.27). Given the above state-space realization of $M(s)$, we seek a check for the condition $\Phi(j\omega) \geq 0$ and thus of the γ-bounded real property. Theoretically, one might imagine checking this property by evaluating $\bar{\sigma}(M(j\omega))$ over the whole frequency domain.

The Bounded Real Lemma.

Suppose one is given a state-space realization of a transfer function matrix $G(s)$; then under certain conditions on the state-space realization, the *bounded real lemma* provides an algebraic test for the bounded real property in terms of an algebraic Riccati equation.

Lemma 5.3.1 *Consider the realization of a transfer function matrix $M(s) = C(sI - A)^{-1}B + D$ such that A is stable and $\bar{\sigma}(M(\infty)) = \bar{\sigma}(D) < \gamma$ (and therefore $U = \gamma^2 I - D^T D > 0$). Then $M(s)$ is γ-bounded real if and only if there exists a strong solution*

$P = P^T$ *of the ARE:*

$$PA + A^T P - C^T C - (PB - C^T D)U^{-1}(PB - C^T D)^T = 0. \qquad (5.29)$$

Proof: See section 6.1 in Chapter 6 where the related theory is reviewed. □

Importantly, the above lemma allows for the case where the infinity norm bound is achieved; i.e. $\|M\|_\infty = \gamma$. In general, it is simpler to solve the ARE when there is a strict bound on the infinity norm.

The task of $\mathcal{H}_\infty$ controller synthesis is basically the following; describe in state space terms control laws $K(s)$ which render $T_{zw}^K(s)M(s)$ γ-bounded real. In general, this is not a trivial question. The strict bounded real lemma has played an important role in the development of the standard suboptimal state-space $\mathcal{H}_\infty$ results (see [85], [109] and [25]). In particular, it is shown in [85] that the algebraic Riccati equation which features in the (strict) bounded real lemma plays a key role in the derivation of the pair of algebraic Riccati equations associated with the existence of $\mathcal{H}_\infty$ control laws. However the strict bound results are not adequate for the multiple objective control problem since they do not allow for the possibility that the specified γ-bound is achieved. Since the strict bounded real lemma plays such a fundamental role in the development of the standard suboptimal $\mathcal{H}_\infty$ control theory, study of the (non-strict) bounded real lemma is likely to facilitate a similar development for $\mathcal{H}_\infty$ control with a non-strict bound; i.e. where $\|T_{zw}^K\|_\infty \leq \gamma$.

It should be noted that a study of the $\mathcal{H}_\infty$ problem with a non-strict bound has been carried out for the full-information case in [38]. As is noted in [38]; "The primary difficulty at optimality is that the Riccati equations need not exist and/or the associated Hamiltonians may have imaginary axis eigenvalues". It is not the purpose here to develop an algorithm for mixed $\mathcal{H}_2$ / $\mathcal{H}_\infty$ synthesis, but to develop a computational tool for solving the algebraic Riccati equation which arises in the bounded real lemma, which is a closely related to the $\mathcal{H}_\infty$ objective. From the point of view of control design, the bounded real lemma is primarily an *analysis* and not explicitly a *synthesis* result. However, it is hoped that an understanding of the algebraic conditions and computational issues involved will facilitate advances in the synthesis theory.

Remarks:

a) Note that the above result is limited in applicability to those transfer function matrices for which we can be sure that $\bar{\sigma}(M(\infty)) < \gamma$. Thus this lemma is not applicable to transfer function matrices which achieve the γ-bound at infinity.

b) Note that it is *not* assumed that (A, B) is controllable.

This is important in the context of control design for the following reason: a state space realization of a closed loop transfer function matrix $T_{zw}^K(s)$ may have stable modes which are uncontrollable via the disturbance $w(t)$. □

Spectral Factorization.

Given a bounded real transfer function $M(s)$, the algebraic Riccati equation of the bounded real lemma is closely related to the *spectral factorization problem* for the spectral matrix $\Phi(s) = \gamma^2 I - M^T(-s)M(s)$. Briefly, given a strong solution of the algebraic Riccati equation, it is possible to construct a square transfer matrix $W(s)$ which is minimum phase (has no zeros in the open right half plane), and satisfies $\Phi(s) = W^T(-s)W(s)$. This connection is reviewed in detail in Chapter 6.

These connections with an algebraic Riccati equation are true for a broader class of spectral factorization problems which includes the bounded real lemma as a special case.

In Chapter 6, spectral matrices are treated which have state-space realizations as follows:

$$\Phi(s) = U + B^T(-sI - A^T)^{-1}V(sI - A)^{-1}B + B^T(-sI - A^T)^{-1}S + S^T(sI - A)^{-1}B \quad (5.30)$$

where each constant matrix is real, $V = V^T$, $U = U^T$, and the following assumptions are made

CA.1 (A, B) is stabilizable.

CA.2 $U > 0$.

The main purpose of Chapter 6 is to develop an algorithm for solving the algebraic Riccati equation associated with the spectral matrix $\Phi(s)$:

$$PA + A^T P + V - (PB + S)U^{-1}(PB + S)^T = 0, \quad (5.31)$$

thus providing a check of the property $\Phi(j\omega) \geq 0$. In particular, we shall be concerned with realizations of spectral matrices which which have imaginary axis invariant zeros. These may correspond with transmission zeros of $\Phi(s)$ and/or nonminimal modes in its realization.

Discussion of the Assumptions on the Spectral Matrix $\Phi(s)$.

Let us now briefly consider the implications of assumptions **CA.1** and **CA.2** in the context of the bounded real lemma and the extent to which they are reasonable in a control design context. Suppose K is a control law which internally stabilizes a generalized plant $G(s)$ and that the closed-loop operator can be realized as $T_{zw}^K = D(K) + C(K)(sI - A(K))^{-1}B_K$ where $\Re\{A(K)\} < 0$. In general, we *cannot* assume that $(A(K), B(K))$ is controllable, however. If we assume that all weighting functions are strictly stable, then we can deduce that $(A(K), B(K))$ must be a *stabilizable* pair.

5.4 Algorithms for solving the ARE in Spectral Factorization.

Recall that the ARE can be used to completely characterize the bounded real constraint. An algorithm which find the strong solution of the ARE when it exists is therefore an analysis tool, in the sense that it allows one to check the $\mathcal{H}_\infty$ constraint. If a strong solution of the ARE can be calculated, then the corresponding closed-loop system is bounded real. On the other hand, we would also like the algorithm to indicate if no such solution exists, indicating that the bounded real constraint is violated. Algorithms for solving AREs which determine whether the *strict* bound is satisfied are widespread and reliable. The aim in the present work is to provide algorithms which solve the ARE when the bounded real constraint is *achieved.*

In Chapter 6, an algorithm is developed for solving the ARE associated with any state-space spectral factorization problem of the form described in (5.30). This includes spectral matrices arising in the bounded real lemma as well as other problems from system and control theory. The algorithm proposed in Chapter 6 hinges on transformation of the continuous time spectral matrix to a discrete time spectral matrix using the bilinear transformation $s = \alpha\frac{z+1}{z-1}$. This is a standard technique for dealing with continuous time problems. An account of the literature in this area is given in Chapter 6. The resulting discrete time spectral factorization problem is equivalent to the continuous time problem in that a strong solution to the continuous time ARE is also a strong solution of the discrete algebraic Riccati equation associated with the discrete time spectral matrix. The discrete time spectral matrix inherits unit circle zeros where the continuous time spectral matrix had imaginary axis zeros.

The proposed solution of the discrete time (and thus the continuous time) spectral factorization problem depends on the convergence of a certain Riccati difference equation to the strong solution of the discrete algebraic Riccati equation. Study of the rate of convergence of this difference equation is the main subject of Chapter 7. The new result established in Chapter 7 is that Riccati difference equations associated with discrete time spectral matrices with unit circle zeros are guaranteed to converge at a rate of at least $\frac{1}{k}$, where k is the iteration number of the Riccati difference equation. In general the RDE will not converge at a rate faster than this either. This is slower than the exponential convergence rate which occurs when there are no unit circle zeros.

In Chapter 6 it is shown that a doubling algorithm is available for the RDE. The doubling algorithm calculates RDE iterates at iterations $k = 2^n$ only, where $n = 1, 2, \ldots$ is the iteration number of the doubling algorithm. The resulting algorithm presented in Chapter 6 therefore has a convergence rate of $\frac{1}{2^n}$. The resulting algorithm is directly applicable to the bounded real problem but can be applied to a broad class of spectral matrices with some straightforward preliminary transformations.

Chapter 6

An Iterative Method for Continuous Time Spectral Factorization with Imaginary Axis Invariant Zeros.

Summary.

Recall that the main objective of $\mathcal{H}_\infty$ control is to achieve a specified bound on the infinity norm of certain closed-loop transfer function matrices whilst maintaining closed-loop stability. In other words, one wishes to design a controller subject to the constraint that certain transfer function matrices be bounded-real. It is well known that the bounded real constraint is equivalent to a related transfer function matrix $\Phi(s) = \Phi^T(-s)$ satisfying the *spectral property* $\Phi(j\omega) \geq 0$ on the imaginary axis. When $\Phi(s)$ has this property, it is called a *spectral matrix*. In Chapter 5, it was argued that the *boundary* of the bounded real constraint is likely to play a critical role in $\mathcal{H}_\infty$ robust controller synthesis with secondary synthesis objectives. In other words, the infinity norm of the closed loop system is very likely to *achieve* the specified bound. The spectral matrix associated with the bounded real constraint in this case has at least one imaginary axis invariant zero. Realizations of spectral matrices which have imaginary axis invariant zeros can arise in other contexts also.

A class of state space realizations of continuous time spectral matrices is defined in this chapter which includes, but is broader than, those which arise in considering the bounded real constraint. The class of spectral matrices treated has been made more general due to the anticipated utility of the results in other design contexts, some of which are summarized in Appendix G.1. It is well known that the spectral property is equivalent to the existence of a so-called strong solution of an algebraic Riccati equation (ARE) associated with the state space realization of the spectral matrix. The main aim of this chapter is to develop an algorithm which calculates the strong solution of the ARE when it exists. Such algorithms effectively provide a means of checking whether the spectral property holds and enable the state space construction of a *spectral factorization* of $\Phi(s)$. Most standard solution techniques for the ARE fail when the spectral matrix

has imaginary axis invariant zeros. This is a key deficiency which this chapter seeks to address.

The algorithm for solving the algebraic Riccati equation which is proposed relies on three key facts. Firstly, it is a standard result that a bilinear transformation of the Laplace variable can be used to transform a continuous time spectral matrix into a discrete time spectral matrix. The associated discrete spectral factorization problem is equivalent to the continuous time problem in the following sense: a solution to the continuous time ARE is also a solution to a discrete algebraic Riccati equation (DARE) associated with the discrete spectral matrix. Secondly, iterates of a Riccati difference equation (RDE) associated with the discrete time problem converge to the strong solution of the algebraic Riccati equation at a known rate (a result established in Chapter 7). Thirdly, established doubling algorithms for LQ control and Kalman filtering problems are shown in this chapter to be applicable to the RDEs associated with the more general discrete time spectral factorization problem. The doubling algorithm enables a considerable improvement in the convergence rate of the RDE and thus of the overall algorithm.

6.1 A Review of Continuous Time State-Space Spectral Factorization.

A fairly general class of realizations of spectral matrices is introduced. The assumptions imposed on these realizations concern stabilizability of the underlying linear system and nonsingularity of the spectral matrix at infinity. This class of spectral matrices includes those arising in the context of the bounded real lemma, as well as those arising in a number of other important systems and control problems.

The role of the linear matrix inequality and the algebraic Riccati equation in the state-space solution of the spectral factorization problem is reviewed next. Such state space spectral factorization results have been under development for some thirty years (see e.g. [1]). However under the particular assumptions made here on the state space realization of the spectral matrix, it is only comparatively recently that results have become available in the literature. The primary aim of this section is to summarize a number of those results which have direct relevance to the problem at hand.

As far as the development of this chapter is concerned, the key conclusion of the review contained in this section is Theorem 6.1.1, which states that the spectral property is equivalent to the existence of the so-called strong solution of an algebraic Riccati equation.. In terms of the bounded real lemma and therefore the $\mathcal{H}_\infty$ objective, the main conclusion is that the desired closed-loop property $\|T_{zw}^K\|_\infty \leq \gamma$, which is essentially a frequency domain constraint, is equivalent to the existence of a strong solution of an algebraic Riccati equation. Techniques for finding such solutions are the subject of later sections in this chapter.

6.1.1 Continuous Time Spectral Factorization.

Spectral Matrices and their Factorization.

For the purpose of this presentation, a *spectral matrix* $\Phi(s)$ in continuous time is a square matrix of real rational functions of $s \in \mathbb{C}$ which satisfies $\Phi^T(-s) = \Phi(s)$ and for which

$$\Phi(j\omega) \geq 0 \tag{6.1}$$

for all $\omega \in \mathbb{R} \cup \{\infty\}$. Henceforth, for the sake of brevity, we refer to the above inequality as the *spectral property.* Note that $\Phi(s)$ may have unbounded entries at a finite number of points on the imaginary axis. We shall be interested in spectral matrices for which $\Phi(\infty)$ is nonsingular. A consequence of the latter assumption is that $\det\{\Phi(s)\} \not\equiv 0$, we then say that $\Phi(s)$ is (generically) *nonsingular.* Another way of expressing this is that $\Phi(s)$ has full normal rank.

Any real rational transfer function matrix $W(s)$ which satisfies $\Phi(s) = W^T(-s)W(s)$ is called a *spectral factor* of the spectral matrix $\Phi(s)$. Spectral factors are not unique and need not be square. However, here we limit our search to those spectral factors which are *minimal rank* in the sense that they have full normal row rank. Such spectral matrices are unique up to left multiplication by a unitary matrix (see [106]). When $\Phi(s)$ is nonsingular, such spectral factors are square and are of prime importance in furnishing solutions to the many engineering problems which can be posed in terms of spectral matrices (see appendix G.1 for a summary of a number of these problems). Of particular importance in such contexts are so-called *minimum phase* spectral factors, which have the property that $W^{-1}(s)$ is analytic when $\Re\{s\} > 0$. It is often desirable that $W(s)$ itself be stable, however this is not stipulated in the state-space treatment of spectral factorization which we now summarize.

State-Space Realizations of Spectral Matrices.

In this work, we consider rational spectral matrices which can be realized as follows:

$$\begin{aligned}\Phi(s) &= U + B^T(-sI - A^T)^{-1}V(sI - A)^{-1}B + B^T(-sI - A^T)^{-1}S + S^T(sI - A)^{-1}B \\ &= U + \begin{pmatrix} S^T & -B^T \end{pmatrix} \begin{pmatrix} sI - A & 0 \\ -V & sI + A^T \end{pmatrix}^{-1} \begin{pmatrix} B \\ S \end{pmatrix} \end{aligned} \tag{6.2}$$

where each constant matrix is real, $V = V^T$, $U = U^T$, and the following assumptions are made:

CA.1 (A, B) is stabilizable.

CA.2 $U > 0$.

Invariant Zeros of the Spectral Matrix Realization.

Of central interest in the present chapter is the role of the invariant zeros of the realization of $\Phi(s)$. Recall that the invariant zeros of the realization given in (6.2) are the values of $s \in \mathbb{C}$ for which the following matrix pencil is less than its normal rank:

$$\mathcal{Z}(s) = \begin{pmatrix} sI - A & 0 & B \\ -V & sI + A^T & S \\ S^T & -B^T & U \end{pmatrix}. \tag{6.3}$$

It should be noted that invariant zeros may occur due to the nonminimality of the realization of the spectral matrix or due to transmission zeros (see the discussion of invariant zeros of a realization of a transfer function matrix in the summary of **Notation, Definitions and Fundamental Results** at the beginning of this thesis).

Spectral Matrix Realization for the Bounded Real Lemma.

We now investigate some of the implications of the assumptions **CA.1** and **CA.2** in treating bounded real matrices. Suppose one is given a realization of a strictly γ-bounded real[1] matrix $M(s) = C(sI-A)^{-1}B+D$, then the associated spectral matrix $\Phi(s) = \gamma^2 I - M^T(-s)M(s)$ has a realization as given in (6.2) with $U = \gamma^2 I - D^T D$, $V = -C^T C$ and $S = -C^T D$. Assumption **CA.2** is satisfied, provided $\bar{\sigma}(D) < \gamma$. Assumption **CA.1** says that all unstable modes of (A, B) must be controllable. Suppose that $M(s) = T_{zw}^K(s)$ is a closed loop matrix, mapping disturbances to objective signals, which results from implementing an internally stabilizing control law on a generalized plant. If we consider only stable weighting functions, A is stable and (A, B) is trivially stabilizable. Note that in many applications, not all modes of the closed loop system will necessarily be controllable via B however.

Relationship with a Linear Quadratic Optimal Control Problem.

Another informative way of expressing realizations of spectral matrices as given in (6.2) is as follows;

$$\Phi(s) = \begin{pmatrix} B^T(-sI - A^T)^{-1} & I \end{pmatrix} \begin{pmatrix} V & S \\ S^T & U \end{pmatrix} \begin{pmatrix} (sI - A)^{-1}B \\ I \end{pmatrix}. \tag{6.4}$$

This expression reveals a connection with the following linear quadratic optimal control problem (with indefinite cost).

Subject to the state-space dynamics

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), & (6.5) \\ x(0) &= x_0, & (6.6) \end{aligned}$$

choose the control input $u(t)$ ($t \in [0, \infty)$) to minimize the quadratic cost function

$$J(u(t), x_0) = \int_{t=0}^{\infty} \begin{pmatrix} x^T(t) & u^T(t) \end{pmatrix} \begin{pmatrix} V & S \\ S^T & U \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} dt. \tag{6.7}$$

[1] Recall from section 5.3 of Chapter 5 that $M(s)$ is said to be strictly γ-bounded real if it is stable and $\|M\|_\infty < \gamma$.

Note that it has *not* been assumed here (or in any of the above discussion in this chapter) that the cost matrix $\begin{pmatrix} V & S \\ S^T & U \end{pmatrix}$ is nonnegative definite. The LQ optimal control problem with sign indefinite cost matrix has been extensively studied in connection with spectral factorization and the Riccati equation (see for example [103]). In [8], an indefinite optimal control problem is treated in a study of the positive real lemma (see item **c)** in appendix G.1 for a description of the spectral matrix arising in this case). In both of the above references, connections are established between the existence of solutions to the above optimal control problem and the spectral property.

Remarks on Assumptions CA.1 and CA.2 :

a) Assumption **CA.2** is a sufficient condition for a spectral matrix with realization (6.2) to be (generically) nonsingular.

b) As commented earlier, in the case of the bounded real lemma (A, B) may have stable uncontrollable modes. Thus it is important that the assumption **CA.1** is made and not the stronger assumption that (A, B) is controllable.

c) Whilst the earliest work on state space spectral factorization was done under the assumption that (A, B) is controllable, a number of problems have been treated in the literature in which this assumption has been relaxed, thus allowing for the possibility of stable uncontrollable modes. This has been considered for linear quadratic control in Problem 3.1.1 of [6], in the study of stabilizing solutions of the ARE in [81], more generally in the study of hermitian solutions of the ARE in [30], in the context of a study of state-space algebraic conditions for the spectral property in Theorem 7.1 of [18], and in [20] where a deflation procedure is developed for solving the spectral factorization problem.

d) The assumption adopted in in [20] is that (A, B) be *sign controllable* which is somewhat weaker than **CA.1**. □

6.1.2 Spectral Factorization via a Linear Matrix Inequality.

We next present a lemma which describes transformations which generate a family of alternative realizations of $\Phi(s)$ as given in (6.2) and (6.4). This result provides a foundation for the state-space solution to the spectral factorization problem which is reviewed here.

Alternative Realizations of the Spectral Matrix.

Lemma 6.1.1 *Let $\Phi(s)$ be a nonsingular spectral matrix with a realization given in (6.4). Let $P = P^T$ be any real symmetric matrix, then an alternative realization of $\Phi(s)$ is*

$$\Phi(s) = \begin{pmatrix} B^T(-sI - A^T)^{-1} & I \end{pmatrix} \begin{pmatrix} PA + A^T P + V & PB + S \\ (PB + S)^T & U \end{pmatrix} \begin{pmatrix} (sI - A)^{-1}B \\ I \end{pmatrix}. \tag{6.8}$$

Proof: This result can be verified using standard manipulations. See for example [103].
□

Remark: Consider the linear quadratic optimal control problems associated with the family of realizations introduced in the above lemma. The state dynamics and quadratic control weighting are unchanged from (6.4), however there is a cancelling redistribution between the quadratic state weighting and the cross-terms. Thus $\Phi(s)$ can be associated with a whole family of different optimal control problems. □

The Linear Matrix Inequality.

With reference to the above lemma, consider the task of finding real symmetric solutions P of the *Linear Matrix Inequality (LMI)*

$$\begin{pmatrix} PA + A^T P + V & PB + S \\ (PB + S)^T & U \end{pmatrix} \geq 0. \tag{6.9}$$

Under the condition that (A, B) is controllable, the existence of real symmetric solutions to the LMI has been related to the existence of infima of the LQ optimal control problem defined in the previous section (again see [103]). In fact solutions of the LMI are closely associated with solutions to the so-called *dissipation inequality* which establishes an infimum for $J(u(t), x_0)$ in the LQ control problem. This connection is not given further attention here.

Note that any symmetric solution P of the LMI allows the spectral matrix $\Phi(s)$ to be factored. Given a solution of (6.9), there are many ways one can factor the left hand side of the LMI, to obtain matrices L and J such that

$$\begin{pmatrix} PA + A^T P + V & PB + S \\ (PB + S)^T & U \end{pmatrix} = \begin{pmatrix} L \\ J^T \end{pmatrix} \begin{pmatrix} L^T & J \end{pmatrix}. \tag{6.10}$$

Clearly then from (6.8), it follows that the matrix $W(s) = J + L^T(sI - A)^{-1}B$ is a spectral factor of $\Phi(s)$.

In summary, given *any* solution of the LMI, it follows that $\Phi(s)$ as realized in (6.4) is a spectral matrix (since a spectral factor can be constructed as described above). The question of under what conditions on the realization of a spectral matrix one can be sure of the existence of a solution to the LMI is much more difficult. It turns out that in order to completely characterize the spectral property in terms of the LMI, it is necessary to focus on a particular class of LMI solutions which we discuss next; those which *minimize the rank* of the LMI.

Rank Minimizing LMI Solutions and the Spectral Property.

It is of importance in many applications to find those solutions of the LMI which are *rank minimizing*, in that they minimize the rank of the left hand side of the LMI (6.10). These solutions are important since they result in *minimal rank* spectral factors. (Recall that minimal rank spectral factors are those with full normal row rank.)

It has been shown in [18] that satisfaction of the spectral property $\Phi(j\omega) \geq 0$ is equivalent to (amongst other things) the existence of a rank-minimizing solution of the LMI. The conditions imposed in [18] are slightly stronger than those imposed here. Whilst assumption **CA.2** is maintained, a slightly stronger assumption than **CA.1** is imposed, being that the realization of $\Phi(s)$ as given in (6.2) has no uncontrollable closed right half plane modes. More recently in [20], under the *weaker* condition than **CA.1**, that (A, B) is *sign controllable*[2], it has been shown that the spectral property is equivalent to the existence of a rank minimizing solution to the LMI. We now summarize a somewhat weaker result which addresses the problem under the assumptions at hand.

Lemma 6.1.2 *Suppose one is given a real rational transfer function matrix $\Phi(s) = \Phi^T(-s)$, realized as in (6.2) such that the assumptions* **CA.1** *and* **CA.2** *hold. Then $\Phi(j\omega) \geq 0$ for all $\omega \in \mathbb{R} \cup \{\infty\}$ if and only if there exists a real, symmetric, rank-minimizing solution P of the LMI:*

$$\begin{pmatrix} PA + A^TP + V & PB + S \\ (PB+S)^T & U \end{pmatrix} \geq 0. \tag{6.11}$$

Proof: The *if* part follows immediately from the previous discussion. The *only if* part is more difficult and has been established in [20]. □

6.1.3 Spectral Factorization via an Algebraic Riccati Equation.

The Algebraic Riccati Equation.

Since by **CA.2**, $\Phi(\infty) = U$ is nonsingular, note that the following inequality follows from the LMI:

$$\begin{pmatrix} I & -(PB+S)U^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} PA + A^TP + V & PB + S \\ (PB+S)^T & U \end{pmatrix} \begin{pmatrix} I & -(PB+S)U^{-1} \\ 0 & I \end{pmatrix}^T$$
$$= \begin{pmatrix} PA + A^TP + V - (PB+S)U^{-1}(PB+S)^T & 0 \\ 0 & U \end{pmatrix} \geq 0. \tag{6.12}$$

The minimum possible rank of the LMI is the normal rank of $\Phi(s)$, which here equals the rank of U. It follows from the above inequality that P is a rank minimizing solution of the LMI if and only if it is a solution of the *algebraic Riccati equation (ARE)*:

$$PA + A^TP + V - (PB+S)U^{-1}(PB+S)^T = 0. \tag{6.13}$$

Note that we consider only real symmetric solutions of the ARE. Given any such P, it then follows immediately from (6.8) and (6.12) that

$$\Phi(s) = \left(I + B^T(-sI - A^T)^{-1}(PB+S)U^{-1}\right) U \left(I + U^{-1}(PB+S)^T(sI-A)^{-1}B\right) \tag{6.14}$$

[2]The pair (A, B) is said to be *sign controllable* if for any uncontrollable mode λ, the matrix $(A - sI \;\; B)$ has full rank for either $s = \lambda$ or $s = -\bar{\lambda}$.

and therefore that $W(s) = U^{\frac{1}{2}} + U^{-\frac{1}{2}}(PB+S)^T(sI-A)^{-1}B$ is a minimal rank spectral factor of $\Phi(s)$.

Remark: Note that $W(s)$ is constructed using the two matrices A and B which appear in the state variable realization (6.2) of the spectral matrix $\Phi(s)$. Observe therefore that such spectral factors inherit their poles from the dynamics of the associated optimal control problem. This is an important feature in many applications of spectral factorization. □

Maximal and Strong Solutions of the ARE.

A solution P of the ARE is called *maximal* if $P - \tilde{P} \geq 0$, for any other symmetric solution $\tilde{P}$ of the ARE. We restrict our consideration here to only the class of real symmetric solutions, for which maximal solutions, when they exist, must be *unique* (see [30]). The following result is an immediate consequence of a more general result stated in Theorem 2.1 of [30] on the existence of maximal solutions.

Lemma 6.1.3 *Suppose the matrices $\{A, B, V, U, S\}$ are real with $U = U^T > 0$, $V = V^T$ and (A, B) stabilizable. Suppose the following ARE has at least one real symmetric solution*

$$PA + A^T P + V - (PB+S)U^{-1}(PB+S)^T = 0. \tag{6.15}$$

It follows that there exists a maximal real symmetric solution P of the ARE. Moreover, it satisfies

$$\Re\{\lambda_i(A - BU^{-1}S^T - BU^{-1}B^T P)\} \leq 0. \tag{6.16}$$

Remarks:

a) Note in particular that the spectral property does *not* play a role in this result since the class of AREs considered includes but is not restricted to those arising in spectral factorization problems.

b) In the context of the LQ optimal control problem defined in 6.1.1, it is well known that by choosing P to be the maximal solution of the ARE one can minimize the cost function $J(x_0, u(\cdot))$ via the state feedback control law $u(t) = -U^{-1}\left(S^T + B^T P\right)x(t)$ (see e.g. [30]). When this control law is implemented on the system $\dot{x}(t) = Ax(t) + Bu(t)$, the *closed loop matrix* $A - BU^{-1}S^T - BU^{-1}B^T P$ describes the resulting dynamics. □

With reference to the above lemma, we call a solution of (6.13) for which $\Re\{\lambda_i(A - BU^{-1}S^T - BU^{-1}B^T P)\} \leq 0$ a *strong* solution. Under the assumption **CA.1** (that (A, B) is stabilizable), it can be shown (see Appendix H) that whenever such a P exists, it is unique.

By inverting the state space realization of the spectral factor $W(s)$ constructed from a strong solution P, it can be easily deduced that the eigenvalues of the above closed-loop matrix are the invariant zeros of $W(s)$ since:

$$W^{-1}(s) = U^{-\frac{1}{2}} - U^{-1}(PB+S)^T \left(sI - (A - BU^{-1}S^T - BU^{-1}B^TP)\right)^{-1} BU^{-\frac{1}{2}}. \quad (6.17)$$

Thus, minimum phase spectral factors can be found using the strong solution of the ARE.

The Spectral Property and Strong Solutions of the ARE.

To summarize the results which have been reviewed in this section, we now state an algebraic condition which is equivalent to the (frequency domain) spectral property $\Phi(j\omega) \geq 0$. This theorem is a straightforward consequence of results available in the literature. Observe that imaginary axis invariant zeros of the spectral matrix are not precluded in the statement of this theorem.

Theorem 6.1.1 *Suppose one is given a real rational transfer function matrix $\Phi(s) = \Phi^T(-s)$, realized as in (6.2) such that the assumptions* **CA.1** *and* **CA.2** *hold. Then $\Phi(j\omega) \geq 0$ for all $\omega \in \mathbb{R} \cup \{\infty\}$ if and only if there exists a strong solution $P = P^T$ of the ARE*

$$PA + A^TP + V - (PB+S)U^{-1}(PB+S)^T = 0. \quad (6.18)$$

Proof: The *if* part follows immediately from the previous discussion in this subsection. The *only if* part is somewhat more difficult. Firstly, note from Lemma 6.1.2 that the spectral property is equivalent to the existence of at least one rank minimizing solution to the LMI, which we call $\tilde{P}$. Secondly, since $U > 0$, it follows from the arguments at the beginning of this subsection that $\tilde{P}$ is a solution of the ARE. Thirdly, by application of Lemma 6.1.3, the ARE has a maximal solution P, which is unique and must be strong. □

To summarize, the main conclusion of this section is that, under the assumptions **CA.1** *and* **CA.2** *on the realization of the spectral matrix, one need consider only the strong solution of the ARE in characterizing the spectral property in state space terms. Firstly, if we can find a strong solution of the ARE, then the spectral property is satisfied. On the other hand, if a strong solution of the ARE does not exist, then the spectral property does not hold.*

The Bounded Real Lemma.

An immediate consequence of this result is the bounded real lemma, which plays an important role in describing in state space terms the bound on the $\mathcal{H}_\infty$ norm of a transfer function matrix.

Lemma 6.1.4 *Consider the realization of a transfer function matrix $M(s) = C(sI - A)^{-1}B + D$ such that A is stable and $\bar{\sigma}(M(\infty)) = \bar{\sigma}(D) < \gamma$ (and therefore $U = \gamma^2 I - D^TD > 0$). and $U = \gamma^2 I - D^TD > 0$. Then $M(s)$ is γ-bounded real if and only if there* *exists a strong solution $P = P^T$ of the ARE:*

$$PA + A^T P - C^T C - (PB - C^T D)U^{-1}(PB - C^T D)^T = 0. \tag{6.19}$$

Proof: See subsection 6.1.3 in Chapter 6. □

Note that the conditions adopted in the above lemma allow for the possibility that the transfer function matrix achieves the γ-bound at finite frequencies. In section 5.2 of Chapter 5, it was shown how multiple objective $\mathcal{H}_\infty$ robust control problems can give rise to closed loop transfer functions which have this property. However, it should be noted that the conditions adopted in Lemma 6.1.4 preclude spectral matrices which achieve the γ-bound at infinity.

6.2 Invariant Subspace Algorithms for Solving the ARE.

The main conclusion of the previous section is that the *existence* of a strong solution to the algebraic Riccati equation is a necessary and sufficient condition for satisfaction of the spectral property $\Phi(j\omega) \geq 0$. As is described in Chapter 5, algorithms for calculating strong solutions of the ARE are of likely utility in the development of multiple objective $\mathcal{H}_\infty$ robust controller synthesis algorithms. Moreover, the fact that strong solutions of the ARE enable state space construction of spectral factors is of significance in many advanced control design methodologies. Reliable algorithms for the computation of ARE solutions are an essential element in realizing the potential benefits of these methodologies. A deficiency in commercially available software is that it is generally not capable of solving the Riccati equation when the spectral matrix has imaginary axis invariant zeros.

The purpose of the present section is to review briefly a number of different approaches to constructing strong solutions of the ARE. We take particular interest in the ability of the algorithms to accommodate realizations of spectral matrices which have imaginary axis invariant zeros. Each approach is based on well known insights which relate the strong solution of the ARE to a particular invariant subspace of a so-called Hamiltonian matrix (which can be constructed directly from the problem data). A number of algorithms based on explicit computation of a basis for this invariant subspace are discussed. These algorithms can be applied when the spectral matrix has imaginary axis eigenvalues; however a number of complexities arise in such cases. Another invariant-subspace type algorithm which we examine in closer detail is based on application of the *matrix sign iteration* to the Hamiltonian matrix. Whilst this algorithm is *not* able to accommodate cases where the spectral matrix has imaginary axis invariant zeros, it is simpler, more straightforward to implement and more amenable to parallel implementation than other invariant subspace type algorithms.

Whilst the main algorithm introduced in later in this chapter for solving AREs is *not* an invariant subspace based algorithm, it is similar to the matrix sign algorithm in that it is *iterative* and straightforward to implement. Importantly, the algorithm it is directly applicable to cases where the spectral matrix has imaginary axis invariant zeros.

6.2.1 Solution via Hamiltonian Matrix Invariant Subspaces.

Perhaps the most common methods for solving the ARE (6.13) involve an invariant subspace calculation for a so-called *Hamiltonian* matrix which can be constructed from the matrices in the realization (6.2) of $\Phi(s)$.

Since U is assumed nonsingular, a realization of the inverse of $\Phi(s)$ can be constructed explicitly from (6.2) as follows:

$$\Phi^{-1}(s) = U^{-1} - U^{-1} \begin{pmatrix} S^T & B^T \end{pmatrix} (sI - H)^{-1} \begin{pmatrix} B \\ S \end{pmatrix} U^{-1} \tag{6.20}$$

where

$$H = \begin{pmatrix} A - BU^{-1}S^T & -BU^{-1}B^T \\ -V + SU^{-1}S^T & -A^T + SU^{-1}B^T \end{pmatrix} \in \mathbb{R}^{n\times n} \tag{6.21}$$

where is a Hamiltonian[3] matrix and n is the dimension of the (square) matrix A. Note that it follows from (6.20) that the invariant zeros of $\Phi(s)$ are the eigenvalues of H.

The basis of many approaches to solving the ARE is the fact that there is a one-to-one correspondence between solutions of (6.13) and certain invariant subspaces of the Hamiltonian matrix H. Let the matrices $X, Y \in \mathbb{R}^{n\times n}$ together constitute a basis for an n-dimensional invariant subspace of H:

$$H \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix} \Lambda, \tag{6.22}$$

The square matrix $\Lambda \in \mathbb{R}^{n\times n}$ has eigenvalues equal to those of the Hamiltonian matrix as an operator restricted to the invariant subspace described by X and Y. It is a well-known result that there is a one to one correspondence between solutions P of the ARE and H-invariant subspaces having the form: $\mathrm{Im}\left\{\begin{pmatrix} I \\ P \end{pmatrix}\right\}$. Thus if X and Y together describe a basis for an invariant subspace of H and if X is invertible, then $P = YX^{-1}$ is a solution of the ARE. In order to ensure *symmetry* of P, it is also necessary that the matrices X and Y describing the invariant subspace satisfy the additional property $YX^{-1} = X^{-T}Y^T$. If $\Re\{\lambda_i(\Lambda)\} < 0$, it happens that this property is automatic. However, when Λ has imaginary axis eigenvalues this symmetry property will not necessarily hold. For a full discussion and account of the literature in this area, see [49] and [54].

Since the strong solution of (6.13) is unique, there is a *unique* choice of Hamiltonian invariant subspace associated with it (see [55], [49], [30] and [18]). By right-multiplying (6.22) by X^{-1}, it is easy to show that

$$A - BU^{-1}S^T - BU^{-1}B^T P = X\Lambda X^{-1}. \tag{6.23}$$

[3]For a definition, see the collection of **Notation, Definitions and Fundamental Results** at the beginning of this thesis.

It follows that the choice of invariant subspace corresponding to the strong solution consists of two components. The first component is the stable invariant subspace of the Hamiltonian matrix of dimension n_s. The second component is a particular invariant subspace of H which corresponds to imaginary axis eigenvalues and has has dimension $n - n_s$. (Note that this invariant subspace is a subspace of the largest invariant subspace of H corresponding to imaginary axis eigenvalues which has dimension $2(n - n_s)$.)

If the stable invariant subspace of H has dimension $n_s < n$, the unstable invariant subspace also has dimension n_s. This is due to the fact that if H is Hamiltonian, $-\bar{\lambda}_i(H)$ is always an eigenvalue of H if $\lambda_i(H)$ is. The dimension of the imaginary axis invariant subspace is $2(n - n_s)$, however the required imaginary axis invariant subspace has dimension $(n - n_s)$ only.

Invariant Subspace Calculation via QR-type Algorithms.

If the Hamiltonian matrix has no imaginary axis eigenvalues, determination of the strong solution is straightforward, as it only requires calculation of the stable invariant subspace of the Hamiltonian. In this case, the strong solution of (6.13) is actually *stabilizing*, i.e., $\Re\{\lambda_i(A - BU^{-1}S^T - BU^{-1}B^TP)\} < 0$. Solution techniques for this case are well developed and reliable algorithms can be found in commercial software packages. One well known and reliable means for computing a basis for the stable invariant subspace employs a QR algorithm for calculating an ordered Schur decomposition of the Hamiltonian matrix H (see [59]). The result of this algorithm is a basis for the stable invariant subspace, from which the stabilizing solution of the ARE can be immediately calculated. See also [58] for a survey of various numerical methods which are applicable when the Hamiltonian matrix has no imaginary axis eigenvalues.

Here we take particular interest in the case when the Hamiltonian matrix *does* have imaginary axis eigenvalues. Recall that imaginary axis spectral matrix transmission zeros arise when considering the boundary of the γ-bounded real constraint in the context of multiple objective robust control. Reliable computational schemes for handling such cases are scarce. Invariant subspaces of H corresponding to imaginary axis eigenvalues are always of even dimension (a fact which was elucidated in [18]) and must be split into half (one half becoming a part of the invariant subspace which determines P). This splitting must take place in a manner which ensures that the Riccati equation solution is symmetric; i.e. X and Y must be chosen such that $YX^{-1} = X^{-T}Y^T$. Obviously this cannot be done simply on the basis of the eigenvalue, which is possible when the eigenvalues are not on the imaginary axis. At present, most commercially available software is not capable of correctly splitting the subspace corresponding to imaginary axis eigenvalues.

To accomplish the correct splitting of the imaginary axis invariant subspace, it is critical to take into account the Hamiltonian structure in solving the eigenvalue problem for H. So-called structure-preserving QR algorithms have been developed in which symplectic orthogonal transformations are employed to find the relevant subspace (see

[18] for a summary of results in this area). A matrix $M \in \mathbb{R}^{2n\times 2n}$ is called *symplectic* if it satisfies the identity $M^T J M = J$, where

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}. \tag{6.24}$$

Such symplectic transformations when applied to H preserve its Hamiltonian structure at each QR step. It should also be noted that orthogonal transformations are particularly attractive from a numerical point of view.

In [18], the Hamiltonian eigenstructure has been revealed in detail. A concept for an algorithm for finding ARE solutions is also presented in [18]. This algorithm proceeds via QR-type iterations and deflations but is not a Hamiltonian QR algorithm. Hamiltonian structure is not preserved at each iteration, but is first destroyed and then reconstructed component by component. This algorithm has been implemented in software by the first author of [18] and has proven numerically reliable and efficient.

A Deflation Procedure for the ARE.

An algorithm for calculating the ARE solution which accommodates imaginary axis invariant zeros has recently been proposed in [20]. The algorithm differs in that it does not aim to explicitly calculate a basis for the invariant subspace associated with the strong solution. Instead the strong solution is constructed piece by piece and at each iteration a new reduced-dimension problem is created by a deflation procedure. This approach is attractive since it reveals the detailed structure of the ARE solution explicitly. However, the numerical properties of this procedure are not likely to be as attractive as the QR-type invariant subspace techniques which use orthogonal basis transformations.

Iterative Techniques.

The structured QR algorithm approach, described in [18] and in the references therein, is not addressed in detail in this thesis. The aim here rather is to find an *iterative* method of solution which does not require the sequence of basis transformations which is generally involved in a QR-type approach. Rather than element-by-element manipulation of matrices with piece-by-piece construction of the invariant subspace, iterative methods produce a *sequence* of matrices, which *at each iteration* can be used to find an approximate solution to the Riccati equation. In the next subsection, we review the *matrix sign iteration*, an iterative algorithm which is known to be very effective in solving the ARE, provided the Hamiltonian matrix does not have imaginary axis eigenvalues. The main purpose of the present chapter is to develop an iterative algorithm which *is* applicable in such cases and which also has known convergence properties.

6.2.2 ARE Solution via the Matrix Sign Iteration.

An effective solution procedure for the ARE, which is based on application of the matrix sign iteration to the Hamiltonian matrix, has been demonstrated in [90] and [28].

However, this approach is guaranteed to work only when the Hamiltonian matrix has no imaginary axis eigenvalues. This approach does not rely on explicit calculation of the invariant subspace corresponding to the ARE solution; however the connection between Hamiltonian matrix invariant subspaces and Riccati equation solutions which was described in the previous subsection is implicit in the technique. We first review the fundamental concepts and main results related to this approach.

The Scalar Sign Function

We define the *sign function* of a scalar variable $x \in \mathbb{C}$ as

$$\operatorname{sgn}(x) = \begin{cases} +1 & \Re\{x\} > 0 \\ -1 & \Re\{x\} < 0 \end{cases} . \tag{6.25}$$

Note that this function is *not* well defined on the imaginary axis. The sign function can be calculated by applying a standard Newton iteration to the function $f(z) = z^2 - 1$. This iteration has initial condition $z_0 = x \in \mathbb{C}$ and reads

$$z_{n+1} = \frac{1}{2}\left(z_n + z_n^{-1}\right). \tag{6.26}$$

Provided $\Re\{x\} \neq 0$, this iteration will always converge quadratically to one of its fixed points, ± 1, whichever corresponds to the sign of x.

The Matrix Sign Function.

The matrix generalization of the sign function, introduced by Roberts [90], is defined in the following manner. Suppose $M \in \mathbb{R}^{p \times p}$ has Jordan form $M = T(D+N)T^{-1}$ where D is a diagonal matrix with the eigenvalues of M, $\{\lambda_1, \ldots, \lambda_p\}$, on its diagonal and N a nilpotent matrix with entries only on the first upper off-diagonal; then the *matrix sign* of M, $Z = \operatorname{sgn}(M)$ is defined as

$$Z = T \operatorname{diag}(\operatorname{sgn}(\lambda_1), \ldots, \operatorname{sgn}(\lambda_p))\ T^{-1}. \tag{6.27}$$

Thus the matrix Z is only defined when all eigenvalues of H lie *off* the imaginary axis. Its eigenvalues are all ± 1 and it satisfies $Z^2 = I$. It can be shown (see [90]) that when it exists, the matrix sign function of $M \in \mathbb{R}^{p \times p}$ is the limit of the sequence of matrices generated using the iteration below, with initial condition $Z_0 = M$:

$$Z_{n+1} = \frac{1}{2}\left(Z_n + Z_n^{-1}\right). \tag{6.28}$$

In fact, it has recently been shown in [44] that this iteration is just one of a whole class of rational iterative methods that calculate the matrix sign function.

Riccati Equation Solution via the Matrix Sign Function.

Application of the matrix sign iteration to a Hamiltonian matrix H having no imaginary axis eigenvalues effects a partitioning of invariant subspaces, corresponding to its left and right half-plane eigenvalues. This fact is used to establish the following lemma.

Lemma 6.2.1 *Let $H \in \mathbb{R}^{2n\times 2n}$ be the Hamiltonian matrix in (6.21) and suppose it has no imaginary axis eigenvalues. Denote*

$$\operatorname{sgn}(H) = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix},$$

with $Z_{ij} \in \mathbb{R}^{n\times n}$, $i,j \in \{1,2\}$. If it exists, the unique strictly stabilizing solution P of (6.13) is a solution of the linear equation:

$$\begin{pmatrix} Z_{12} \\ Z_{22}+I \end{pmatrix} P = -\begin{pmatrix} Z_{11}+I \\ Z_{21} \end{pmatrix}. \tag{6.29}$$

Proof: This is proved in [41] and relies on the fact that if a stabilizing solution to the Riccati equation exists, the Hamiltonian matrix can be block diagonalized with strictly stable closed-loop eigenvalues in the first diagonal block and unstable eigenvalues in the second. Application of the sign iteration to this diagonalized matrix reveals the result. □

Remarks:

a) One advantage of the matrix sign algorithm over QR type invariant subspace algorithms is that it can be applied directly to the Hamiltonian matrix without having to carry out basis transformations. Only matrix inversions and additions are required.

b) It has also been shown recently that the matrix sign algorithm is amenable to parallel implementation (see [29], [58]), unlike the direct QR-type algorithms for which parallel implementations appear to be difficult to find. This is an important advantage for AREs which arise in problems involving dynamical systems of very high order.

c) The matrix sign algorithm as presented above is usually scaled to improve numerical accuracy. See for example [58] and also [45] which contains a detailed account of the convergence rates of the sign iteration. □

6.3 Limits to ARE Solution Accuracy.

Before proceeding, it is instructive to consider how we are to measure the *quality* of any approximate solution to the ARE which we might calculate. Suppose there exists a genuine solution P of the ARE (6.13) which we seek to calculate by some as yet unspecified algorithm. One measure of the quality of an approximate solution $\hat{P}$ of (6.13) is the *residual*

$$\mathcal{R}(\hat{P}) = \hat{P}A + A^T\hat{P} + V - (\hat{P}B+S)U^{-1}(\hat{P}B+S)^T. \tag{6.30}$$

However, it has been shown that this is not always a good indicator of solution quality [42]. This can be seen by observing that a computed solution $\hat{P}$ with small nonzero residual is an exact solution to a perturbed ARE:

$$A^T\hat{P} + \hat{P}A - (\hat{P}B+S)U^{-1}(\hat{P}B+S)^T + V + \hat{\epsilon}N = 0. \tag{6.31}$$

This equation is simply a re-writing of equation (6.30) where, without loss of generality, $\hat{\epsilon}N = -\mathcal{R}(\hat{P})$ with $N \in \mathbb{R}^{n \times n}$ a nonzero matrix with unity norm and $\hat{\epsilon} = \|\mathcal{R}(\hat{P})\|$, a real constant which is obviously small if the residual is small.

Whilst the norm of the residual might be quite small, the norm of the error in the Riccati equation solution $\|P - \hat{P}\|$ can be very large, as we now show. Consider a family of AREs which are related to the perturbed ARE (6.31) and which are parametrised by $\epsilon \in \mathbb{R}$;

$$A^T P_\epsilon + P_\epsilon A - (P_\epsilon B + S)U^{-1}(P_\epsilon B + S)^T + V + \epsilon N = 0 \tag{6.32}$$

Note that when $\epsilon = 0$, P solves this ARE and that when $\epsilon = \hat{\epsilon}$, $\hat{P}$ is a solution. The question we would like to answer is: how sensitive is P_ϵ to variations in ϵ? Supposing for the moment that the derivative $\frac{\partial P_\epsilon}{\partial \epsilon}$ exists at $\epsilon = 0$, then differentiation of (6.32) with respect to ϵ reveals that

$$(A - BU^{-1}S^T - BU^{-1}B^T P)^T \left.\frac{\partial P_\epsilon}{\partial \epsilon}\right|_{\epsilon=0} + \left.\frac{\partial P_\epsilon}{\partial \epsilon}\right|_{\epsilon=0} (A - BU^{-1}S^T - BU^{-1}B^T P) + N = 0. \tag{6.33}$$

Observe that when the closed loop matrix $(A - BU^{-1}S^T - BU^{-1}B^T P)$ has imaginary axis eigenvalues, $\frac{\partial P_\epsilon}{\partial \epsilon}$ is not well defined at $\epsilon = 0$, except for nongeneric N and even then it is not computable from (6.33). It may be that this derivative is very large close to $\epsilon = 0$ (which is likely to be the case if $(A - BU^{-1}S^T - BU^{-1}B^T P_\epsilon)$ has stable eigenvalues, some very close to the imaginary axis); or if the derivative at $\epsilon = 0$ does not exist.

It can be seen that the difference between the actual solution of the ARE and the candidate solution

$$\hat{P} - P \approx \hat{\epsilon} \left.\frac{\partial P_\epsilon}{\partial \epsilon}\right|_{\epsilon=0} \tag{6.34}$$

can be large, even when the residual itself is quite small. In conclusion, finding a solution of (6.13) is made more difficult because of the *bad conditioning* of the problem when imaginary axis Hamiltonian eigenvalues are present (for a discussion of this concept, see [60]). This difficulty is intrinsic to the problem we wish to solve and imposes a fundamental limitation to any proposed algorithm.

6.4 Discrete Time State-Space Spectral Factorization Review.

In this section, we review a number of results for a class of discrete time state space spectral factorization problems. It will be discussed in section 6.5 how a continuous time spectral matrix with a state space description as described in section 6.1 can be transformed into a discrete-time spectral matrix which has a realization of the type described in the present section. The class of discrete spectral matrices introduced in this section include not only those which arise from such transformations of continuous time spectral matrices, but covers a broad class which is of significance in other contexts, including discrete LQ control and filtering for example.

The discussion here can be viewed partly in parallel with the discussion on continuous time spectral factorization in section 6.1. The emphasis is mainly on the relationship between the discrete linear matrix inequality and the discrete algebraic Riccati equation, which becomes important in the next subsection. Connections between the discrete spectral property and the existence of so-called *strong* solutions of the discrete ARE are not discussed in detail. Such connections do exist but are not of immediate relevance to the present chapter.

6.4.1 Discrete Time Spectral Factorization.

Discrete Time Spectral Matrices and their Factorization.

In discrete time, a *spectral matrix* $\Psi(z)$ is a square real rational matrix-valued function of a complex variable z with the properties that $\Psi^T(z^{-1}) = \Psi(z)$, and $\Psi(e^{j\theta}) \geq 0$ for all $\theta \in [0, 2\pi)$. We consider only spectral matrices which are generically *nonsingular*, in the sense that $\det(\Psi(z)) \not\equiv 0$. Like their continuous time counterparts, discrete time spectral matrices arise naturally in the description of stationary stochastic processes, the discrete bounded-real and positive-real lemmas and in the formulation of linear control and filtering problems. The construction of spectral factors is central to the solution of the abovementioned problems. A *spectral factor* $\Omega(z)$ of $\Psi(z)$ is a real rational matrix-valued function of the complex variable z which satisfies $\Omega^T(z^{-1})\Omega(z) = \Psi(z)$. If, in addition, $\Omega^{-1}(z)$ exists and is analytic when $|z| > 1$, $\Omega(z)$ is called a *minimum phase* spectral factor. It is well known (see Theorem 1 of [80] and Theorem 4.1 in Chapter 9 of [7]) that if $\Psi(z)$ is nonsingular, there exists a spectral decomposition of the form

$$\Psi(z) = \Theta^T(z^{-1})N\,\Theta(z) \tag{6.35}$$

where N is a symmetric positive definite matrix and $\Theta(z)$ is a square real rational transfer function matrix which is invertible, satisfies $\Theta(\infty) = I$ and, along with its inverse, is analytic when $|z| > 1$. Hence a minimum-phase spectral factor of $\Psi(z)$ exists and can be constructed as $\Omega(z) = N^{\frac{1}{2}}\Theta(z)$.

State Space Realizations of Spectral Matrices.

Let $\Psi(z)$ be a generically nonsingular spectral matrix which can be realized as follows:

$$\begin{aligned}\Psi(z) &= \hat{U} + \hat{G}^T(z^{-1}I - \hat{F}^T)^{-1}\hat{S} + \hat{S}^T(zI - \hat{F})^{-1}\hat{G} + \hat{G}^T(z^{-1}I - \hat{F}^T)^{-1}\hat{V}(zI - \hat{F})^{-1}\hat{G} \\ &= \hat{U} + \begin{pmatrix} \hat{S}^T & \hat{G}^T \end{pmatrix} \begin{pmatrix} zI - \hat{F} & 0 \\ -\hat{V} & z^{-1}I - \hat{F}^T \end{pmatrix}^{-1} \begin{pmatrix} \hat{G} \\ \hat{S} \end{pmatrix}\end{aligned} \tag{6.36}$$

where each constant matrix is real, $\hat{V} = \hat{V}^T$ and $\hat{U} = \hat{U}^T$ and where we make the following assumptions:

DA.1 $(\hat{F}, \hat{G})$ is stabilizable.

DA.2 $\hat{U}$ is nonsingular.

Spectral matrices with the above state-space structure arise in Kalman filtering, linear optimal control, the discrete bounded real lemma and in realization theory for stationary discrete stochastic processes.

Invariant Zeros of the Spectral Matrix Realization.

The nonzero invariant zeros of the realization of a discrete spectral matrix as given in (6.2) are the values of $z \in \mathbb{C}$ for which the following matrix is less than its normal rank, [19]:

$$\hat{\mathcal{Z}}(z) = \begin{pmatrix} zI - \hat{F} & 0 & \hat{G} \\ -\hat{V} & z^{-1}I - \hat{F}^T & \hat{S} \\ \hat{S}^T & \hat{G}^T & \hat{U} \end{pmatrix}. \tag{6.37}$$

As in the continuous time case, invariant zeros may occur due to the nonminimality of the realization of the spectral matrix or due to transmission zeros (i.e. values of $\lambda \in \mathbb{C}$ such that $\text{rank}\{\Psi(\lambda)\} < \text{normrank}\{\Psi(z)\}$). Whilst in the continuous time case, invariant zeros can be calculated as the eigenvalues of an associated Hamiltonian matrix, this is not true in general in the discrete time case. However, calculation of invariant zeros in discrete time (including any at $z = 0$) can be reduced to the calculation of the generalized eigenvalues of a related matrix pencil (see e.g. [19]).

Relationship with a Discrete Linear Quadratic Optimal Control Problem.

Note the following informative re-expression of the discrete time spectral matrix $\Psi(z)$:

$$\Psi(z) = \begin{pmatrix} \hat{G}^T(z^{-1}I - \hat{F}^T)^{-1} & I \end{pmatrix} \begin{pmatrix} \hat{V} & \hat{S} \\ \hat{S}^T & \hat{U} \end{pmatrix} \begin{pmatrix} (zI - \hat{F})^{-1}\hat{G} \\ I \end{pmatrix}. \tag{6.38}$$

This expression reveals a close connection with the following discrete time linear quadratic optimal control problem.

Subject to the state-space dynamics

$$x_{k+1} = Ax_k + Bu_k, \tag{6.39}$$

$$x_{k=0} = x_0, \tag{6.40}$$

choose the control input u_k *(* $k \in \{0, 1, 2, \ldots\}$ *) to minimize the quadratic cost function*

$$J(u_k, x_0) = \sum_{k=0}^{\infty} \begin{pmatrix} x_k^T & u_k^T \end{pmatrix} \begin{pmatrix} \hat{V} & \hat{S} \\ \hat{S}^T & \hat{U} \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix}. \tag{6.41}$$

Unless explicitly stated, it will not be assumed that the matrix $\begin{pmatrix} \hat{V} & \hat{S} \\ \hat{S}^T & \hat{U} \end{pmatrix}$ is non-negative definite. For an account of the connection between this optimal control problem and discrete time spectral factorization, see [80].

Remarks on Assumptions DA.1 and DA.2 :

a) Assumption **DA.1** includes a broad class of practically significant state-space realizations of discrete spectral matrices.
b) Assumption **DA.1** also holds for the class of discrete time spectral matrices described in section 6.5, which result from application of the bilinear transformation to *continuous* time spectral matrices.
c) It is shown in Lemma I.0.1 of Appendix I that assumption **DA.2** entails no loss of generality, provided the spectral matrix is generically nonsingular. It is shown in the proof of Lemma I.0.1 that if one is given a realization of a nonsingular $\Psi(z)$ which violates **DA.2**, an alternative realization of $\Psi(z)$ can be constructed which satisfies **DA.2**.
□

6.4.2 Spectral Factorization via a Linear Matrix Inequality.

Alternative Realizations of the Discrete Time Spectral Matrix.

Similar to the continuous time case, we now describe a family of alternative realizations of the discrete time spectral matrix $\Psi(z)$ which plays an important role in the development of state space algorithms for spectral factorization.

Lemma 6.4.1 *Let $\Psi(z)$ be a nonsingular spectral matrix with a realization given in (6.38) where each constant matrix is real and $\hat{V} = \hat{V}^T$, $\hat{U} = \hat{U}^T$. Let $P = P^T$ be any real symmetric matrix, then an alternative realization of $\Psi(z)$ is*

$$\Psi(z) = \begin{pmatrix} \hat{G}^T(z^{-1}I - \hat{F}^T)^{-1} & I \end{pmatrix} \begin{pmatrix} \hat{V} - P + \hat{F}^T P \hat{F} & \hat{S} + \hat{F}^T P \hat{G} \\ (\hat{S} + \hat{F}^T P \hat{G})^T & \hat{U} + \hat{G}^T P \hat{G} \end{pmatrix} \begin{pmatrix} (zI - \hat{F})^{-1}\hat{G} \\ I \end{pmatrix} \tag{6.42}$$

Proof: A class of simple basis transformations is now used to show this (standard) result.
Observe that $\Psi(z)$ can also be expressed in the following manner:

$$\begin{aligned} \Psi(z) &= \hat{U} + \mathcal{G}^T \mathcal{F}^{-1}(z) \mathcal{G} && (6.43) \\ &= \hat{U} + \begin{pmatrix} \hat{G}^T & \hat{S}^T \end{pmatrix} \begin{pmatrix} 0 & zI - \hat{F} \\ z^{-1}I - \hat{F}^T & -\hat{V} \end{pmatrix}^{-1} \begin{pmatrix} \hat{G} \\ \hat{S} \end{pmatrix}. && (6.44) \end{aligned}$$

By introducing the basis transformation

$$\mathcal{T}(P) = \begin{pmatrix} I & 0 \\ \hat{F}^T P \hat{G} & I \end{pmatrix} \tag{6.45}$$

one obtains the following new realization of $\Psi(z)$

$$\Psi(z) = \hat{U} + \bar{\mathcal{G}}^T \bar{\mathcal{F}}^{-1}(z) \bar{\mathcal{G}} \tag{6.46}$$

where $\bar{\mathcal{G}}^T = \mathcal{G}^T \mathcal{T}^T(P)$, $\bar{\mathcal{F}}(z) = \mathcal{T}(P)\mathcal{F}(z)\mathcal{T}^T(P)$. Expansion of the identity (6.46) results in the new realization of the spectral matrix given in the lemma statement. □

Remark: Observe that these transformations redistribute the weighting in the performance index of the associated discrete time optimal control problem, whilst preserving the underlying linear system. In contrast with the analogous continuous time result in Lemma 6.1.1, the above alternative realizations of the discrete spectral matrix alter the quadratic weighting on the control input. □

The Discrete Linear Matrix Inequality.

Analogous to the continuous time case, with reference to the above lemma, consider the task of finding real symmetric solutions P to the following *Linear Matrix Inequality (LMI)*:

$$\begin{pmatrix} \hat{V} - P + \hat{F}^T P \hat{F} & \hat{S} + \hat{F}^T P \hat{G} \\ (\hat{S} + \hat{F}^T P \hat{G})^T & \hat{U} + \hat{G}^T P \hat{G} \end{pmatrix} \geq 0. \tag{6.47}$$

It follows immediately from Lemma 6.4.1 that, given such a solution to the LMI, $\Psi(z)$ can be factored; there are any number of constant matrices $\hat{L}$ and $\hat{J}$ such that

$$\begin{pmatrix} \hat{V} - P + \hat{F}^T P \hat{F} & \hat{S} + \hat{F}^T P \hat{G} \\ (\hat{S} + \hat{F}^T P \hat{G})^T & \hat{U} + \hat{G}^T P \hat{G} \end{pmatrix} = \begin{pmatrix} \hat{L} \\ \hat{J}^T \end{pmatrix} \begin{pmatrix} \hat{L}^T & \hat{J} \end{pmatrix}. \tag{6.48}$$

It then follows immediately from (6.42) that the transfer function matrix $\Omega(z) = \hat{J} + \hat{L}^T (zI - \hat{F})^{-1} \hat{G}$ is a spectral factor of $\Psi(z)$.

Rank Minimizing Solutions of the Discrete LMI and the Spectral Property.

Rank minimizing solutions of the discrete LMI are those which minimize the rank of the matrix on the left hand side of (6.47). Rank minimizing solutions of the discrete LMI are of importance since they give rise to spectral factors which are of minimal normal row rank.

Remark: It has been shown quite recently in [19] that, under the assumption that $(\hat{F}, \hat{G})$ is stabilizable and that $\Psi(z)$ is generically nonsingular, there exists at least one rank-minimizing solution of the LMI. For the purpose of this chapter, which is the development of an algorithm for *continuous time* spectral factorization, this result is not of immediate relevance. However, it is likely to be of significance in the treatment of intrinsically discrete spectral factorization problems. For completeness, the result of [19] is now summarized. Suppose one is given a real rational transfer function matrix $\Psi(z) = \Psi^T(z^{-1})$, realized as in (6.36) such that the assumptions **DA.1** and **DA.2** hold. Then $\Phi(e^{j\theta}) \geq 0$ for all $\theta \in [0, 2\pi)$ if and only if there exists a real, symmetric, rank-minimizing solution $\hat{\Phi}$ of the LMI:

$$\begin{pmatrix} \hat{V} - \hat{\Phi} + \hat{F}^T \hat{\Phi} \hat{F} & \hat{S} + \hat{F}^T \hat{\Phi} \hat{G} \\ (\hat{S} + \hat{F}^T \hat{\Phi} \hat{G})^T & \hat{U} + \hat{G}^T \hat{\Phi} \hat{G} \end{pmatrix} \geq 0. \tag{6.49}$$

□

6.4.3 Spectral Factorization via an Algebraic Riccati Equation.

Whilst in the continuous time case, the study of rank minimizing solutions to the LMI leads directly to the study of an associated algebraic Riccati equation, in general an analogous statement cannot always be made in the discrete time case. This is highlighted in [100] where the concept of *strongly rank-minimizing* solutions of the discrete LMI is introduced; a solution $\hat{\Phi}$ of the discrete LMI is called strongly rank-minimizing if the rank of the matrix on the left hand side of the LMI equals that of its $(2,2)$ block; $\hat{U}+\hat{G}^T\hat{\Phi}\hat{G}$. A solution of the LMI must be strongly rank minimizing to also be a solution of the discrete ARE. However, provided $\Psi(z)$ is (generically) nonsingular, no such distinction need be drawn since the following result holds:

Lemma 6.4.2 *Suppose one is given a nonsingular spectral matrix $\Psi(z)$, realized as in (6.36) such that the assumptions* **DA.1** *and* **DA.2** *hold. Then any rank minimizing solution $\hat{\Phi}$ of the associated LMI (6.47) has the following property*

$$N = \hat{U} + \hat{G}^T\hat{\Phi}\hat{G} > 0 \tag{6.50}$$

and it satisfies the discrete algebraic Riccati equation (DARE):

$$\hat{\Phi} = \hat{F}^T\hat{\Phi}\hat{F} + \hat{V} - (\hat{F}^T\hat{\Phi}\hat{G} + \hat{S})(\hat{U} + \hat{G}^T\hat{\Phi}\hat{G})^{-1}(\hat{F}^T\hat{\Phi}\hat{G} + \hat{S})^T. \tag{6.51}$$

Proof: See [100]. □

Such equations arise in many contexts, including spectral factorization [4], and infinite-horizon control and filtering problems [5, 7]. Given a real symmetric solution of the DARE (6.51), a state-space realization of a spectral factor of $\Psi(z)$ can be constructed. It can be demonstrated using (6.51) that (6.35) can be satisfied with the definitions $N = \hat{U} + \hat{G}^T\hat{\Phi}\hat{G}$ and $\Theta(z) = I + N^{-1}(\hat{G}^T\hat{\Phi}\hat{F} + \hat{S}^T)(zI - \hat{F})^{-1}\hat{G}$. This results in a spectral factor

$$\Omega(z) = N^{\frac{1}{2}} + (N^{\frac{1}{2}})^{-T}(\hat{G}^T\hat{\Phi}\hat{F} + \hat{S}^T)(zI - \hat{F})^{-1}\hat{G}, \tag{6.52}$$

where $N^{\frac{1}{2}}$ is a square root of N (not necessarily symmetric).

Strong Solutions to the ARE.

Consider the discrete time linear system associated with the linear quadratic optimal problem defined earlier (in (6.39), (6.40) and (6.41)) with a linear state-feedback control law of the following form: $u_k = -N^{-1}(\hat{G}^T\hat{\Phi}\hat{F} + \hat{S}^T)$. We call the state transition matrix $\tilde{F}$ associated with the closed loop system the *closed-loop matrix*, which is given by:

$$\tilde{F} = \hat{F} - \hat{G}N^{-1}(\hat{G}^T\hat{\Phi}\hat{F} + \hat{S}^T). \tag{6.53}$$

It should be noted that $\tilde{F}$ appears as the state-transition matrix in the following state-space realization of the inverse of the spectral factor $\Omega(z)$ as given in (6.52):

$$\Omega^{-1}(z) = \left(I - N^{-1}(\hat{G}^T\hat{\Phi}\hat{F} + \hat{S}^T)(zI - \tilde{F})^{-1}\hat{G}\right) N^{-\frac{1}{2}}. \tag{6.54}$$

A solution $\hat{\Phi}$ of (6.51) is said to be *strong* if the closed loop matrix $\tilde{F}$ has all its eigenvalues either inside or on the unit circle. Note that the eigenvalues of $\tilde{F}$ are also the invariant zeros of $\Omega(z)$ and thus spectral factors constructed from strong solutions of (6.51) have the minimum phase property. It will be the subject of section 6.6 as to how one can calculate strong solutions of the DARE.

Remarks:

a) Recall that in the continuous time case there exists a close connection between strong and *maximal* solutions of the ARE, as described in Lemma 6.1.3. Note that results concerning the existence or otherwise of *maximal* solutions to the DARE are *not* discussed in detail here. Whilst a study of this question is of interest in its own right, it will become apparent that such results are not required in the present development.

b) The question of under what conditions on the spectral matrix a unique strong solution $\hat{\Phi}$ of the DARE (6.51) exists is not addressed here either. In the later development of a spectral factorization algorithm for continuous time matrices using a related discrete time problem, it will be shown that the existence of a strong solution to the discrete ARE is guaranteed by the existence of a strong solution to the continuous time ARE. Thus when applying the results which have been summarized in the present subsection, it will be possible to assume that such a solution exists for the realization of the spectral matrix at hand.

A number of existence results for strong or stabilizing solutions of the DARE for various classes of spectral matrices are available in the literature. In [80], a class of spectral matrices with $(\hat{F}, \hat{G})$ controllable is considered and it is shown there that, analogous to the continuous time case, the spectral property is a necessary and sufficient condition for existence of a strong solution to the ARE. In [54], a summary of existence results for the case with $(\hat{F}, \hat{G})$ controllable and $\hat{F}$ stable is presented. Conditions for existence and uniqueness of strong solutions to (6.51) have been extensively studied in the context of Kalman filtering and linear optimal control (see [15] and [22] and the references therein). In [22], the condition $(\hat{F}, \hat{G})$ controllable is relaxed to stabilizable. For the case of spectral matrices arising in the discrete bounded real lemma, the question of existence has been addressed in [24] where $(\hat{F}, \hat{G})$ is assumed stabilizable. □

6.5 Connecting Continuous and Discrete Time Spectral Factorization.

In this section, we first review a well known method for constructing a family of discrete time spectral matrices from a continuous time spectral matrix using the bilinear transformation. Each member of the resulting family of discrete time spectral matrices has the property that any solution of the LMI associated with the continuous time spectral matrix is also a solution of the LMI associated with the discrete time problem. The main conclusion of this section is that one can solve for the strong solution of a continuous time ARE by focussing solely on finding the strong solution of any one of the family of discrete AREs associated with the family of spectral matrices. This sets the scene for the remainder of this chapter which is primarily concerned with solving the DARE. The results presented in this section are a minor strengthening of those presented in [4] and draw upon the continuous and discrete time spectral factorization results which were reviewed in sections 6.1 and 6.4.

6.5.1 A Family of Equivalent Discrete Time Spectral Matrices.

Given a spectral matrix $\Phi(s)$ with the realization in (6.2), we describe how the bilinear transformation $s = \alpha\frac{z+1}{z-1}$ gives rise to a family of discrete time spectral matrices:

$$\Psi_\alpha(z) = \Phi(\alpha\frac{z+1}{z-1}). \tag{6.55}$$

After constructing a state-space description of the discrete time spectral matrix $\Psi_\alpha(z)$, the aim is to solve the associated discrete time spectral factorization problem and use the solution of that problem to deduce a solution of the original continuous time problem. This approach is well documented and has been widely applied. See for example the papers [35], [36], [37] and [4] which describe the development and origins of this approach. More recently, similar ideas have been applied in connecting discrete and continuous $\mathcal{H}_\infty$ synthesis problems in [47] and [48].

The first step in finding realizations of the matrices $\Psi_\alpha(z)$ employs a minor variant of a result employed in [37] and [4] which is described in the following lemma.

Lemma 6.5.1 *Suppose one is given the following realization of a continuous time transfer function matrix*

$$G(s) = D + C(sI - A)^{-1}B.$$

Let $\alpha > 0$ denote a real constant which is not an eigenvalue of A. Let the variable $z \in \mathbb{C}$ be defined by the bilinear transformation

$$z = \frac{s+\alpha}{s-\alpha}.$$

In this new variable, $G(s)$ can be expressed as a rational function $\tilde{G}(z)$, having a state-space realization

$$\tilde{G}(z) = G\left(\alpha\frac{z+1}{z-1}\right) = D_\alpha + C_\alpha(zI - A_\alpha)^{-1}B_\alpha \tag{6.56}$$

with constituent matrices given by

$$D_\alpha = D + C(\alpha I - A)^{-1}B, \tag{6.57}$$
$$C_\alpha = -\sqrt{2\alpha}C(\alpha I - A)^{-1}, \tag{6.58}$$
$$B_\alpha = \sqrt{2\alpha}(\alpha I - A)^{-1}B, \tag{6.59}$$
$$A_\alpha = (A + \alpha I)(A - \alpha I)^{-1}. \tag{6.60}$$

Proof: It is easy to show by direct substitution that $\tilde{G}(z) = D + C(\alpha I - A)^{-1}(zI - I)(zI - A_\alpha)^{-1}B$, and from the trivial identity $(zI - I) = (zI - A_\alpha + A_\alpha - I)$, it follows that $\tilde{G}(z) = D_\alpha + C(\alpha I - A)^{-1}(zI - A_\alpha)^{-1}(A_\alpha - I)B$. Note that it is necessary that α *not* be an eigenvalue of A to enable the inverse $(\alpha I - A)^{-1}$ to be calculated. The fact that $A_\alpha - I = 2\alpha(A - \alpha I)^{-1}$ is easy to verify and leads directly to the stated result. □

The next lemma presents state space realizations of the spectral matrix $\Psi_\alpha(z)$, as defined in (6.55). Variations of this result have been proven elsewhere (see for example [4]). In addition to giving state-space realizations, the following lemma highlights the fact that it is easy to ensure that this state-space description of $\Psi_\alpha(z)$ satisfies the assumptions **DA.1** and **DA.2** on discrete time spectral matrices given in section 6.4. This will prove important since we shall later apply discrete time spectral factorization results to this realization of $\Psi_\alpha(z)$ which depend on these assumptions.

Lemma 6.5.2 *Suppose one is given the following realization of a nonsingular continuous-time spectral matrix $\Phi(s)$:*

$$\Phi(s) = \begin{pmatrix} B^T(-sI - A^T)^{-1} & I \end{pmatrix} \begin{pmatrix} V & S \\ S^T & U \end{pmatrix} \begin{pmatrix} (sI - A)^{-1}B \\ I \end{pmatrix}, \tag{6.61}$$

satisfying **CA.1** *(i.e. (A, B) is stabilizable) and* **CA.2** *(i.e. $U > 0$). Let $\alpha > 0$ denote a real constant which is not an eigenvalue of A. Let the variable $z \in \mathbb{C}$ be defined by the bilinear transformation*

$$z = \frac{s + \alpha}{s - \alpha}.$$

In this new variable, $\Phi(s)$ can be expressed as a nonsingular discrete-time spectral matrix $\Psi_\alpha(z) = \Phi\left(\alpha\frac{z+1}{z-1}\right)$, having a state-space realization

$$\Psi_\alpha(z) = \begin{pmatrix} \hat{G}_\alpha^T(z^{-1}I - \hat{F}_\alpha^T)^{-1} & I \end{pmatrix} \begin{pmatrix} \hat{V}_\alpha & \hat{S}_\alpha \\ \hat{S}_\alpha^T & \hat{U}_\alpha \end{pmatrix} \begin{pmatrix} (zI - \hat{F}_\alpha)^{-1}\hat{G}_\alpha \\ I \end{pmatrix}, \tag{6.62}$$

with constituent matrices given by

$$\hat{F}_\alpha = (A + \alpha I)(A - \alpha I)^{-1}, \quad \hat{G}_\alpha = \sqrt{2\alpha}(\alpha I - A)^{-1}B,$$

$$\begin{pmatrix} \hat{V}_\alpha & \hat{S}_\alpha \\ \hat{S}_\alpha^T & \hat{U}_\alpha \end{pmatrix} = \begin{pmatrix} C_\alpha^T & 0 \\ D_\alpha^T & I \end{pmatrix} \begin{pmatrix} V & S \\ S^T & U \end{pmatrix} \begin{pmatrix} C_\alpha & D_\alpha \\ 0 & I \end{pmatrix}, \tag{6.63}$$

where $C_\alpha = -\sqrt{2\alpha}(\alpha I - A)^{-1}$ and $D_\alpha = (\alpha I - A)^{-1}B$.

Moreover, for this realization of $\Psi_\alpha(z)$,

- *$(\hat{F}_\alpha, \hat{G}_\alpha)$ is a stabilizable pair (in the discrete time sense),*
- *$\hat{U}_\alpha$ is generically nonsingular.*

Proof: Consider the following transfer function which plays an important role in the given state-space description of $\Phi(s)$: $W(s) = (sI - A)^{-1}B$. Application of Lemma 6.5.1 to this state-space description yields $W(s) = \tilde{W}(z) = D_\alpha + C_\alpha(zI - \hat{F}_\alpha)^{-1}\hat{G}_\alpha$, with each of the matrices in this realization defined in the lemma statement. Observe that $W^T(-s) = W^T(-\alpha\frac{z^{-1}+1}{z^{-1}-1}) = \tilde{W}^T(z^{-1})$, and that one can therefore write

$$\begin{aligned}
\Psi_\alpha(z) &= \begin{pmatrix} \hat{G}_\alpha^T(z^{-1}I - \hat{F}_\alpha^T)^{-1}C_\alpha & I \end{pmatrix} \begin{pmatrix} V & S \\ S^T & U \end{pmatrix} \begin{pmatrix} C_\alpha(zI - \hat{F}_\alpha)^{-1}\hat{G}_\alpha \\ I \end{pmatrix} \\
&+ D_\alpha^T\left(VC_\alpha(zI - \hat{F}_\alpha)^{-1}\hat{G}_\alpha + S\right) + \left(\hat{G}_\alpha^T(z^{-1}I - \hat{F}_\alpha^T)^{-1}C_\alpha^T + S^T\right)D_\alpha \\
&+ D_\alpha^T V D_\alpha.
\end{aligned}$$

Collecting together all like terms gives the stated result. [4]

We now show that the pair $(\hat{F}_\alpha, \hat{G}_\alpha) = \left((A + \alpha I)(A - \alpha I)^{-1}, \sqrt{2\alpha}(\alpha I - A)^{-1}B\right)$ is stabilizable. Note first that it is a standard result that, provided α is not an eigenvalue of A, then the eigenvalues of $\hat{F}_\alpha$ are $\frac{\lambda(A)+\alpha}{\lambda(A)-\alpha}$. Note that if $\lambda(A)$ is stable (unstable) in a continuous time sense, then the corresponding eigenvalue of $\hat{F}_\alpha$ is stable (unstable) in a discrete time sense. In addition, the corresponding left eigenvectors x^T of A are also left eigenvectors of $\hat{F}_\alpha$. For any left eigenvector x^T of $\hat{F}_\alpha$ corresponding to an unstable eigenvalue, it follows from stabilizability of (A, B) that $x^T\hat{G}_\alpha = \frac{\sqrt{2\alpha}}{\alpha-\lambda(A)}x^T B = 0$.

Note that $\hat{U}_\alpha = D_\alpha^T V D_\alpha + D_\alpha^T S + S^T D_\alpha + U$ is a rational function of α and that $D_{\alpha=\infty} = 0$ whence $\hat{U}_{\alpha=\infty} = U > 0$. It follows immediately that $\hat{U}_\alpha$ as a function of α is generically a nonsingular matrix. □

Remarks:

a) Similar formulae are available for conversion of a discrete time spectral matrix into a continuous time spectral matrix (see e.g. [37]).

b) Suppose that, after carrying out the transformation of a continuous time spectral matrix as described by the above lemma, one obtains a discrete spectral factor $\Omega_\alpha(z)$; then the bilinear transform will give rise to a spectral factor $W(s) = \Omega_\alpha(\frac{s+\alpha}{s-\alpha})$ for the original continuous time spectral matrix. □

6.5.2 Relating Discrete and Continuous Time LMI and ARE Solutions.

A lemma is now presented which shows that a solution of the continuous time LMI is automatically a solution of the discrete LMI associated with the realization of $\Psi_\alpha(z)$ in Lemma 6.5.2. This result is important in that it demonstrates that one can transform to a discrete time problem and do state space calculations which are *directly* relevant to the continuous time case, *without* having to carry out any transformations back to the continuous time representation.

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[4] The author would like to gratefully acknowledge Dr. D.J. Clements who pointed out equation (6.63) in correspondence.

Lemma 6.5.3 *Suppose one is given a realization of a nonsingular continuous-time spectral matrix $\Phi(s)$ as described in Lemma 6.5.2. Let $\alpha > 0$ denote a real constant which is not an eigenvalue of A. Let $\Psi_\alpha(z)$ denote any member of the family of nonsingular discrete time spectral matrices with realizations described in Lemma 6.5.2.*

Suppose the real matrix $P = P^T$ is a solution of the LMI associated with the continuous time spectral matrix $\Phi(s)$;

$$\begin{pmatrix} PA + A^TP + V & PB + S \\ (PB+S)^T & U \end{pmatrix} \geq 0. \tag{6.64}$$

Then P is also a solution to the discrete LMI associated with $\Psi_\alpha(z)$:

$$\begin{pmatrix} \hat{V}_\alpha - P + \hat{F}_\alpha^T P \hat{F}_\alpha & \hat{S}_\alpha + \hat{F}_\alpha^T P \hat{G}_\alpha \\ (\hat{S}_\alpha + \hat{F}_\alpha^T P \hat{G}_\alpha)^T & \hat{U}_\alpha + \hat{G}_\alpha^T P \hat{G}_\alpha \end{pmatrix} \geq 0. \tag{6.65}$$

Proof: Let P be a solution to the continuous LMI as described in the lemma statement. Recall from Lemma 6.1.1 that the associated continuous time spectral matrix $\Phi(s)$ can be realized as in described in (6.8). Application of the bilinear transformation described in Lemma 6.5.2 to this new realization of $\Phi(s)$ results in a new state space realization of $\Psi_\alpha(z)$. Note in particular that the part of this realization which is given by application of (6.63) to the new realization of $\Phi(s)$ is

$$\begin{pmatrix} C_\alpha^T & 0 \\ D_\alpha^T & I \end{pmatrix} \begin{pmatrix} PA + A^TP + V & PB + S \\ (PB+S)^T & U \end{pmatrix} \begin{pmatrix} C_\alpha & D_\alpha \\ 0 & I \end{pmatrix} \geq 0. \tag{6.66}$$

Here C_α and D_α are the same as given in the statement of Lemma 6.5.2. Note that the expression in (6.66) must be nonnegative definite since P is a solution to the continuous time LMI. Standard manipulations of this expression results in the discrete time LMI in (6.65). □

The most important consequence of the above result here concerns solutions of the algebraic Riccati equations associated with rank minimizing solutions of the LMI (6.65).

Theorem 6.5.1 *Suppose one is given a realization of a nonsingular continuous-time spectral matrix $\Phi(s)$ as described in Lemma 6.5.2. Let $\alpha > 0$ denote a real constant which is not an eigenvalue of A. Let $\Psi_\alpha(z)$ denote any member of the family of nonsingular discrete time spectral matrices with realizations described in Lemma 6.5.2. Suppose P is a strong solution to the continuous-time ARE*

$$PA + A^TP + V - (PB+S)U^{-1}(PB+S)^T = 0, \tag{6.67}$$

then $\hat{\Phi}_\alpha = P$ is a strong solution of the following DARE

$$\hat{\Phi}_\alpha = \hat{F}_\alpha^T \hat{\Phi}_\alpha \hat{F}_\alpha + \hat{V}_\alpha - (\hat{F}_\alpha^T \hat{\Phi}_\alpha \hat{G}_\alpha + \hat{S}_\alpha)(\hat{U} + \hat{G}_\alpha^T \hat{\Phi}_\alpha \hat{G}_\alpha)^{-1}(\hat{F}_\alpha^T \hat{\Phi}_\alpha \hat{G}_\alpha + \hat{S}_\alpha)^T. \tag{6.68}$$

Proof: This result was proven in [4] for the case of spectral matrices arising in the positive real lemma. The proof below proceeds in two steps. Firstly it is shown that P

solves the discrete ARE. Secondly it is shown that it is indeed a strong solution.

By hypothesis, P is a solution of the continuous time ARE. Recall from the discussion in subsection 6.1.3 that since $U > 0$, any solution P of the continuous time ARE is also a rank minimizing solution of the continuous time LMI given in (6.9) and (6.64). Next observe that it is fairly easy to establish that, for a given P, the ranks of the left hand sides of the continuous and discrete LMIs (6.64) and (6.65) are equal (this actually follows from the identity (6.66) connecting the left hand sides of the two inequalities, which is established in the proof of Lemma 6.5.3). Since P is a *rank minimizing* solution of the continuous LMI, it follows that it is simultaneously a rank minimizing solution of the discrete LMI (6.65). Because $\Psi_\alpha(z)$ is generically nonsingular, and the realization of $\Psi_\alpha(z)$ given in Lemma 6.5.2 satisfies all the assumptions in Lemma 6.4.2, one can apply that lemma to to deduce that P must also be a solution of the discrete ARE and that

$$\hat{N}_\alpha = \hat{U}_\alpha + \hat{G}_\alpha^T P \hat{G}_\alpha > 0. \tag{6.69}$$

As the final step in the proof, we now seek to show that P is a strong solution of the DARE (6.68). The objective is thus to show that the following matrix has eigenvalues all with modulus less than or equal to unity:

$$\tilde{F}_\alpha = \hat{F}_\alpha - \hat{G}_\alpha \hat{N}_\alpha^{-1}(\hat{G}_\alpha^T P \hat{F}_\alpha + \hat{S}_\alpha^T). \tag{6.70}$$

By hypothesis, P is a strong solution of the continuous time ARE (6.67). In other words, the following matrix has all its eigenvalues in the closed left half plane:

$$\tilde{A} = A - BU^{-1}S^T - BU^{-1}B^T P. \tag{6.71}$$

Note the following identity

$$\tilde{F}_\alpha = (\tilde{A} + \alpha I)(\tilde{A} - \alpha I)^{-1}, \tag{6.72}$$

which can be proven using the formulae for the state-space matrices of $\Psi_\alpha(z)$ as given in Lemma 6.5.2 (a shortened proof of this identity is presented below). Observe that the matrix $(\tilde{A}+\alpha I)(\tilde{A}-\alpha I)^{-1}$ has eigenvalues in the closed unit disk. It follows immediately from (6.72) that P is a strong solution of the discrete ARE.

Proof of the identity (6.72):
Recall that P can be used to construct a minimum phase spectral factor of $\Phi(s)$:

$$W(s) = U^{\frac{1}{2}} + U^{-\frac{1}{2}}(PB + S)^T(sI - A)^{-1}B. \tag{6.73}$$

Application of the bilinear transformation to $W(s)$ using the transformation formulae in Lemma 6.5.1 gives rise to a spectral factor of the discrete time spectral matrix $\Psi_\alpha(z)$:

$$\Omega_\alpha(z) = W(\alpha\frac{z+1}{z-1}) = \hat{N}_\alpha^{\frac{1}{2}} + (\hat{N}_\alpha^{\frac{1}{2}})^{-T}(\hat{G}_\alpha^T P \hat{F}_\alpha + \hat{S}_\alpha^T)(zI - \hat{F}_\alpha)^{-1}\hat{G}_\alpha \tag{6.74}$$

where it can be checked that

$$\hat{N}_{\alpha}^{\frac{1}{2}} = U^{\frac{1}{2}} + U^{-\frac{1}{2}}(PB+S)^T(\alpha I - A)^{-1}B \tag{6.75}$$

is indeed a square root of $\hat{N}_{\alpha}$ (not symmetric in general). Also note that one can check that

$$(\hat{N}_{\alpha}^{\frac{1}{2}})^{-T}(\hat{G}_{\alpha}^T P \hat{F}_{\alpha} + \hat{S}_{\alpha}^T) = -\sqrt{2\alpha}U^{-\frac{1}{2}}(PB+S)^T(\alpha I - A)^{-1}. \tag{6.76}$$

Observe that the matrix $\tilde{A}$ appears in the inverse of the spectral factor $W(s)$:

$$W^{-1}(s) = U^{-\frac{1}{2}} - U^{-1}(PB+S)^T(sI - \tilde{A})^{-1}BU^{-\frac{1}{2}}. \tag{6.77}$$

Application of the bilinear transformation to this realization using the formulae in Lemma 6.5.1 can be used to show that

$$W^{-1}(\alpha\frac{z+1}{z-1}) \;=\; D_{W^{-1}} + C_{W^{-1}}(zI - A_{W^{-1}})^{-1}B_{W^{-1}}, \tag{6.78}$$

where $D_{W^{-1}} = U^{-\frac{1}{2}} - U^{-1}(PB+S)^T(\alpha I - \tilde{A})^{-1}BU^{-\frac{1}{2}}$, $C_{W^{-1}} = \sqrt{2\alpha}U^{-1}(PB+S)^T(\alpha I - \tilde{A})^{-1}$, $A_{W^{-1}} = (\tilde{A}+\alpha I)(\tilde{A}-\alpha I)^{-1}$ and $B_{W^{-1}} = \sqrt{2\alpha}(\alpha I - \tilde{A})^{-1}BU^{-\frac{1}{2}}$. Moreover, it can be shown that

$$W^{-1}(\alpha\frac{z+1}{z-1}) = \Omega_{\alpha}^{-1}(z), \tag{6.79}$$

where $\Omega_{\alpha}^{-1}(z)$ has the realization

$$\Omega_{\alpha}^{-1}(z) = \left(I - \hat{N}_{\alpha}^{-1}(\hat{G}_{\alpha}^T P \hat{F}_{\alpha} + \hat{S}_{\alpha}^T)(zI - \tilde{F}_{\alpha})^{-1}\hat{G}_{\alpha}\right)\hat{N}_{\alpha}^{-\frac{1}{2}}. \tag{6.80}$$

In fact, it can be shown that the matrices used to realize $\Omega_{\alpha}^{-1}(z)$ are *identical* to those which appear in the realization (6.79) of $W^{-1}(\alpha\frac{z+1}{z-1})$. The identity (6.72) follows immediately. □

In summary, if a strong solution of the continuous time ARE (6.67) exists, then it must also be a strong solution of the discrete time ARE(6.68). This is a key result as it enables discrete time spectral factorization results to be applied to solve continuous-time spectral factorization problems.

6.6 Calculation of the Strong DARE Solution via a Riccati Difference Equation.

In the previous section we concluded that when a strong solution to the continuous time ARE exists, it will also be the strong solution of a discrete ARE, which can be easily constructed from the state-space data for the continuous time spectral matrix using the formulae given in Lemma 6.5.2. We therefore now focus on finding strong solutions of discrete AREs. Whilst the work in this section and the next was motivated primarily by the observations connecting continuous and discrete spectral matrices, most of the material can be read independently of these observations.

The main technique for solving the discrete ARE which we consider in this chapter draws upon the results which are established in Chapter 7 concerning the convergence of Riccati difference equation (RDE) iterates. In essence it is shown there that iterates of a matrix Riccati difference equation associated with the DARE converge to the strong solution of the DARE at least as fast as $\frac{1}{k}$, where k is the number of iterations. These convergence results hold for Riccati difference equations associated with a broad class of discrete spectral factorization problems.

The Riccati Difference Equation arising in Spectral Factorization.

We now return to considering the general discrete time spectral factorization problem discussed in section 6.4. Recall the class of realizations of nonsingular discrete time spectral matrices $\Psi(z)$ which was discussed in section 6.4:

$$\Psi(z) = \hat{U} + \hat{G}^T(z^{-1}I - \hat{F}^T)^{-1}\hat{S} + \hat{S}^T(zI - \hat{F})^{-1}\hat{G} + \hat{G}^T(z^{-1}I - \hat{F}^T)^{-1}\hat{V}(zI - \hat{F})^{-1}\hat{G}, \tag{6.81}$$

where the following assumptions are made:

DA.1 $(\hat{F}, \hat{G})$ is stabilizable.

DA.2 $\hat{U}$ is nonsingular.

Recall also the discrete algebraic Riccati equation associated with $\Psi(z)$:

$$\hat{\Phi} = \hat{F}^T\hat{\Phi}\hat{F} + \hat{V} - (\hat{F}^T\hat{\Phi}\hat{G} + \hat{S})(\hat{U} + \hat{G}^T\hat{\Phi}\hat{G})^{-1}(\hat{F}^T\hat{\Phi}\hat{G} + \hat{S})^T. \tag{6.82}$$

With reference to this DARE, we consider iteration of the closely related *Riccati difference equation (RDE)*:

$$\hat{\Phi}_{k+1} = \hat{F}^T\hat{\Phi}_k\hat{F} + \hat{V} - (\hat{F}^T\hat{\Phi}_k\hat{G} + \hat{S})(\hat{U} + \hat{G}^T\hat{\Phi}_k\hat{G})^{-1}(\hat{F}^T\hat{\Phi}_k\hat{G} + \hat{S})^T, \tag{6.83}$$

with initial condition $\hat{\Phi}_0$.

Remarks:

a) Suppose $\hat{\Phi}$ is the strong solution of the DARE (6.82); then it is well known, and of considerable importance in what follows, that if the initial condition $\hat{\Phi}_0 \geq \hat{\Phi}$, then $\hat{\Phi}_k \geq \hat{\Phi}$ for all $k \in \{1, 2, \ldots\}$. This result is established in chapter 7 and is stated in item *2* of Lemma 7.3.1.

b) In this presentation, we consider only convergence of the RDE iterates from *above*, i.e. $\hat{\Phi}_0 \geq \hat{\Phi}$. In the case of LQ control and filtering, it is well known that convergence can also occur from *below*, i.e. $\hat{\Phi}_0 < \hat{\Phi}$ (see e.g. [15]). Convergence from below for RDEs associated with the general spectral factorization problem cannot be guaranteed in general.

c) The RDE is closely related to a finite horizon LQ optimal control problem defined on the spectral matrix data, which is in fact a finite horizon version of the infinite-horizon optimal control problem defined in section 6.4.1. See [5] for a description of the role of the RDE in the finite horizon LQ problem. □

Convergence of RDE Iterates to the Strong Solution.

An observation which is fundamental to the method we propose here, is that in many circumstances the RDE iterates $\hat{\Phi}_k$ *converge* to the strong solution $\hat{\Phi}$. This fact was recognized some time ago as being of utility in finding solutions to the discrete ARE. For example, in [37], [9], [15] and [22] this convergence is discussed for Kalman filtering and linear quadratic regulator problems. In [4] it is shown that this convergence property can be used to solve the DARE corresponding to a spectral matrix which is the Hermitian part of a discrete positive real matrix. For a summary of related convergence results in the literature, see section 7.1 of Chapter 7. The following result, which is established in Chapter 7, is a fairly straightforward generalization of well known convergence results.

Suppose one is given a discrete time spectral matrix $\Psi(z)$ with realization (6.81) satisfying assumptions **DA.1** *and* **DA.2** *, along with the strong solution $\hat{\Phi}$ of the associated algebraic Riccati equation (6.82). Suppose that the RDE (6.83) has an initial condition which satisfies $\hat{\Phi}_0 \geq \hat{\Phi}$, then*

$$\lim_{k \to \infty} \hat{\Phi}_k = \hat{\Phi}. \tag{6.84}$$

If this approach is to be readily applicable, however some understanding of the convergence *rate* of the RDE iterates is essential. The presence or otherwise of unit circle invariant zeros of the spectral matrix has considerable impact on this rate. The main contribution of Chapter 7 is to establish convergence *rates* of the RDE when the spectral matrix has unit circle invariant zeros. We now summarize some results about convergence rates.

RDE Convergence Rate with no Unit Circle Invariant Zeros.

Suppose one is given a discrete time spectral matrix $\Psi(z)$ with realization (6.81) satisfying assumptions **DA.1** *and* **DA.2** *, along with the strong solution $\hat{\Phi}$ of the associated algebraic Riccati equation (6.82). Suppose also that $\Psi(z)$ has no invariant zeros on the unit circle. If the RDE (6.83) has an initial condition which satisfies $\hat{\Phi}_0 \geq \hat{\Phi}$, then it is well known (see for example [13] and [4]) that RDE iterates converge exponentially to $\hat{\Phi}$; there exist constants $A, \kappa_3 > 0$ such that $1 > \kappa_3$ and*

$$\lambda_{\max}(\hat{\Phi}_k - \hat{\Phi}) \leq A\kappa_3^k. \tag{6.85}$$

This exponential convergence result suggests direct application of the RDE in solving the DARE. Successful application of this approach to the linear quadratic regulator problem and a detailed discussion of computational aspects are reported in [37].

RDE Convergence Rate with Unit Circle Invariant Zeros.

In what follows we allow for the possibility that the discrete spectral matrix has unit circle invariant zeros. Note that if one is given a continuous time spectral matrix which has imaginary axis invariant zeros, the discrete spectral matrices resulting from the transformation of section 6.5 will always have unit circle invariant zeros. Discrete spectral matrices with unit circle invariant zeros arise in other contexts also, some of which are described in more detail in subsection 7.1.2 of Chapter 7. The convergence of RDE iterates has been shown to occur at a $\frac{1}{k}$ rate for some scalar examples in which unit circle invariant zeros are present (see subsection 7.1.3). However, no statement or proof of this result for the general case could be found in the literature. This is the main subject of chapter 7, where the following result is established: (see Theorem 7.5.1)

Suppose one is given a discrete time spectral matrix $\Psi(z)$ with realization (6.81) satisfying assumptions **DA.1** *and* **DA.2**, *along with the strong solution $\hat{\Phi}$ of the associated algebraic Riccati equation (6.82). Suppose also that $\Psi(z)$ has at least one invariant zero on the unit circle. Suppose that the RDE (6.83) has an initial condition which satisfies $\hat{\Phi}_0 \geq \hat{\Phi}$, then there exist constants κ_1, κ_2 (depending on the realization of $\Psi(\cdot)$ and on $\hat{\Phi}_0$) with $\kappa_1 > \kappa_2 > 0$ such that:*

Worst Case Convergence Rate

For all $\epsilon > 0$, there exists an integer k_ϵ such that when $k \geq k_\epsilon$ iterates of (6.83) satisfy

$$\lambda_{\max}(\hat{\Phi}_k - \hat{\Phi}) \leq \frac{\kappa_1 + \epsilon}{k}. \tag{6.86}$$

Best Case Convergence Rate

If the initial condition also satisfies $\hat{\Phi}_0 > \hat{\Phi}$, then for all $\eta \in (0, \kappa_2)$, there exists a k_η such that when $k \geq k_\eta$

$$\lambda_{\max}(\hat{\Phi}_k - \hat{\Phi}) \geq \frac{\kappa_2 - \eta}{k}. \tag{6.87}$$

Compared with the exponential rate when no unit circle invariant zeros are present, the above convergence rates are slow. On their own, these results actually suggest that direct application of the RDE (as in [37] or [4]) to find solutions of the DARE is *not* a good idea when the spectral matrix has unit circle invariant zeros. The iterates are likely to take some time to come within the vicinity of $\hat{\Phi}$ and once they do, numerical errors are not likely to be attenuated well by the algorithm and may overpower the calculation. In this chapter, we shall nevertheless draw upon the $\frac{1}{k}$ convergence result in deriving an algorithm for solving the discrete time ARE. We shall do so in conjunction with a *doubling algorithm* for the RDE which is introduced in the next section. The resulting algorithm has a considerably accelerated convergence rate.

Remark: In the above, we assume *exact computations*. It should be noted that the influence of numerical errors (due to finite wordlength effects) on the accuracy of the final result will depend on the above convergence rate. □

6.7 Doubling Algorithms for RDEs in Spectral Factorization.

In the previous section it became apparent that due to the $\frac{1}{k}$ convergence rate, the need for an alternative to direct iteration of the RDE is particularly acute when the discrete spectral matrix has unit circle invariant zeros. In the present section, we discuss the applicability of the *doubling iteration* to Riccati Difference equations arising in discrete spectral factorization problems. The doubling iteration makes possible a method for finding the limiting solution of RDEs which is potentially numerically superior to direct iteration. This is achieved by calculating RDE iterates $\hat{\Phi}_k$ only at iteration numbers k which are integral powers of two (hence the term *doubling*). In the context of the results on RDE convergence summarized in the previous section, the result is an algorithm whose iterates $\hat{\Phi}_{2^n}$ converge to the strong solution of the DARE at a worst-case rate of $\frac{1}{2^n}$.

In [2] and [7] doubling algorithms are presented for Riccati difference equations in the context of the linear optimal regulator problem and Kalman filtering. In [2] and [7], the following assumptions are effectively made on the realization of the discrete spectral matrix (6.81); $\hat{S} = 0$, $\hat{U} > 0$, $\hat{V} \geq 0$, $(\hat{F}, \hat{G})$ is stabilizable and $(\hat{F}, \hat{V}^{\frac{1}{2}})$ is detectable. These assumptions rule out the possibility that the spectral matrix realization has unit circle invariant zeros. Here our objective is to establish the applicability of the doubling algorithm to RDEs associated with a somewhat more general class of spectral matrices than those arising in the class filtering and regulator problems described above. In particular, we would like to establish the applicability of the doubling algorithm in cases where the realization (6.81) of the spectral matrix $\Psi(z)$ has unit circle invariant zeros, whether they be transmission or decoupling zeros.

Assumptions on the Spectral Matrix Realization for the Doubling Algorithm.

It is assumed that a discrete time spectral matrix is given which, in addition to being nonnegative definite on the unit circle, has a realization of the form described in (6.81). We make the following assumptions on this realization in the development which follows:

DA.1 $(\hat{F}, \hat{G})$ is stabilizable.

DA.2 $\hat{U}$ is nonsingular.

DA.4 $\hat{F} - \hat{G}\hat{U}^{-1}\hat{S}^T$ is nonsingular.

Remarks on the Assumptions:

a) Assumptions **DA.1** and **DA.2** are important in that the RDE convergence results summarized in the previous section rest on them.

b) Note that a *temporary* assumption **DA.3** (that $(\hat{F}, \hat{G})$ is controllable) is introduced in the next chapter in order to establish RDE convergence. This assumption plays *no role* at all in the present chapter.

c) Assumption **DA.4** precludes many discrete time spectral factorization problems with time delays in the underlying dynamics. For example, suppose $\hat{S} = 0$ in (6.81), then **DA.4** precludes the existence of delays which would require $\hat{F}$ to be nonsingular.

d) Note that one does *not* need to assume **DA.4** to establish the RDE convergence results established in Chapter 7. This assumption relates specifically to the development of the doubling algorithm presented in this section.

e) For the purpose of solving continuous time problems via the discrete time problems described in section 6.5, assumption **DA.4** is not restrictive since the parameter α connecting the discrete and continuous time problems can generally always be chosen to ensure **DA.4**. This issue will be explored further in section 6.8. □

Applicability of Doubling Algorithms in Spectral Factorization.

A detailed discussion of doubling algorithms and their utility in finding the limiting solution to Riccati difference equations in Kalman filtering can be found in section 6.7 of [7]. We now recount some of that discussion with some minor modifications which demonstrate how the doubling approach applies to the more general case of spectral factorization at hand. In particular, the spectral property $\Psi(e^{j\theta}) \geq 0$ plays an important role in the proofs presented here.

A Linear Matrix Difference Equation underlying the RDE.

The key observation behind the doubling approach (the proof of which is algebraically intricate but in principle straightforward) concerns the underlying *linear* structure associated with the Riccati difference equation:

$$\hat{\Phi}_{k+1} = \hat{F}^T\hat{\Phi}_k\hat{F} + \hat{V} - (\hat{F}^T\hat{\Phi}_k\hat{G} + \hat{S})(\hat{U} + \hat{G}^T\hat{\Phi}_k\hat{G})^{-1}(\hat{F}^T\hat{\Phi}_k\hat{G} + \hat{S})^T. \tag{6.88}$$

Lemma 6.7.1 *Suppose one is given a discrete-time spectral matrix $\Psi(z)$, with a realization given in (6.81) which satisfies assumptions* **DA.1** *,* **DA.2** *and* **DA.4** *. Suppose $\hat{\Phi}$ is the strong solution of the discrete algebraic Riccati equation (6.82) associated with this realization. Let $\hat{\Phi}_0 \geq \hat{\Phi}$ be an initial condition of the associated Riccati difference equation (6.88).*

Consider the following linear difference equation for $X_k, Y_k \in \mathbb{R}^{n\times n}$ defined for $k \geq 0$:

$$\begin{pmatrix} X_{k+1} \\ Y_{k+1} \end{pmatrix} = \Gamma \begin{pmatrix} X_k \\ Y_k \end{pmatrix} \tag{6.89}$$

with initial condition

$$\begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} = \begin{pmatrix} I \\ \hat{\Phi}_0 \end{pmatrix}, \tag{6.90}$$

and where the linear transition matrix $\Gamma \in \mathbb{R}^{2n\times 2n}$ *is defined as*

$$\Gamma = \begin{pmatrix} \alpha_0^{-1} & \alpha_0^{-1}\beta_0 \\ \gamma_0\alpha_0^{-1} & \alpha_0^T + \gamma_0\alpha_0^{-1}\beta_0 \end{pmatrix} \tag{6.91}$$

with $\alpha_0 = \hat{F} - \hat{G}\hat{U}^{-1}\hat{S}^T$, $\beta_0 = \hat{G}\hat{U}^{-1}\hat{G}^T$ *and* $\gamma_0 = \hat{V} - \hat{S}\hat{U}^{-1}\hat{S}^T$. *Then the following hold true:*

1. X_k *is nonsingular for all* $k \geq 0$.

2. *Iterates of the Riccati difference equation (6.88) can be constructed as*

$$\hat{\Phi}_k = Y_k X_k^{-1}.$$

Proof: When $k = 0$, both *1* and *2* follow trivially from the definitions of X_0 and Y_0 given in (6.90). Suppose next that *1* and *2* hold true for some k. We now show their validity for $k+1$, thus proving the lemma by induction.

Proof of item 1 for $k+1$*:*
The following identity follows from expansion of (6.89) and the propositions that X_k is invertible and that $\hat{\Phi}_k = Y_k X_k^{-1}$:

$$X_{k+1} = (\hat{F} - \hat{G}\hat{U}^{-1}\hat{S}^T)^{-1}(I + \hat{G}\hat{U}^{-1}\hat{G}^T\hat{\Phi}_k)X_k. \tag{6.92}$$

Observe that the first and third terms of the above expression are invertible; $(\hat{F} - \hat{G}\hat{U}^{-1}\hat{S}^T)$ by **DA.4** and X_k by hypothesis. Next we show that under the assumptions stated in the lemma, the second term in (6.92), $(I+\hat{G}\hat{U}^{-1}\hat{G}^T\hat{\Phi}_k)$, is also invertible. Recall from Lemma 6.4.2 that since the spectral matrix $\Psi(z)$ is nonsingular, it follows that $N = \hat{U} + \hat{G}^T\hat{\Phi}\hat{G} > 0$. Since $\hat{U}$ is invertible, this implies the invertibility of $I + \hat{G}^T\hat{\Phi}\hat{G}\hat{U}^{-1}$. Observe next that $I + \hat{G}^T\hat{\Phi}_k\hat{G}\hat{U}^{-1} = I + \hat{G}^T\hat{\Phi}\hat{G}\hat{U}^{-1} + \hat{G}^T\hat{\Delta}_k\hat{G}\hat{U}^{-1}$ where $\hat{\Delta}_k = \hat{\Phi}_k - \hat{\Phi}$. Due to the spectral property $\Psi(e^{j\theta}) \geq 0$, it follows that one can apply item *2* in Lemma 7.3.1 of Chapter 7 to show that $\hat{\Delta}_k \geq 0$, for all $k \geq 1$ (since $\hat{\Delta}_0 \geq 0$). It follows therefore that $I + \hat{G}^T\hat{\Phi}_k\hat{G}\hat{U}^{-1}$ is invertible for all $k \geq 0$. By the matrix inversion lemma $I + \hat{G}^T\hat{\Phi}_k\hat{G}\hat{U}^{-1}$ is invertible if and only if $(I + \hat{G}\hat{U}^{-1}\hat{G}^T\hat{\Phi}_k)$ is invertible. Invertibility of X_{k+1} follows immediately from (6.92).

Proof of item 2 for $k+1$*:*
First note the following easily verified equalities:

$$\begin{aligned} X_{k+1}^{-1} &= X_k^{-1}(I + \beta_0\hat{\Phi}_k)^{-1}\alpha_0, && (6.93)\\ Y_{k+1} &= \left(\gamma_0\alpha_0^{-1}(I + \beta_0\hat{\Phi}_k) + \alpha_0^T\hat{\Phi}_k\right)X_k. && (6.94)\end{aligned}$$

It follows easily from these equalities that

$$Y_{k+1}X_{k+1}^{-1} = \gamma_0 + \alpha_0^T\hat{\Phi}_k(I + \beta_0\hat{\Phi}_k)^{-1}\alpha_0. \tag{6.95}$$

Recall that $\beta_0 = \hat{G}\hat{U}^{-1}\hat{G}^T$ and note that the term whose inverse appears in the equation

(6.95) can be re-expressed as follows using the matrix inversion lemma:

$$\left(I+\hat{G}\hat{U}^{-1}\hat{G}^T\hat{\Phi}_k\right)^{-1}=I-\hat{G}(\hat{G}^T\Phi_k\hat{G}+\hat{U})^{-1}\hat{G}^T\hat{\Phi}_k. \tag{6.96}$$

Note that the inverse in the right hand side of the above equation is guaranteed to exist by virtue of the spectral property, as was explained above in the proof of item *1* of this lemma. Next substitute (6.96) into the equation (6.95) to obtain:

$$Y_{k+1}X_{k+1}^{-1}=\gamma_0+\alpha_0^T\left(\hat{\Phi}_k-\hat{\Phi}_k\hat{G}(\hat{G}^T\hat{\Phi}_k\hat{G}+\hat{U})^{-1}\hat{G}^T\hat{\Phi}_k\right)\alpha_0, \tag{6.97}$$

where α_0 and γ_0 are given in the lemma statement. Subsequent lengthy but straightforward manipulations of the last expression (see for example the proof of Theorem 3 in [37]) can be used to show that the right hand side of (6.97) in fact equals the right hand side of the RDE (6.88). □

Remarks:

a) We call Γ in the above lemma the *linear transition matrix* associated with the RDE (6.88). It is an easily verified and standard result that Γ is a symplectic matrix.

b) The linear matrix difference equation described in the above lemma is directly related to the state/co-state dynamics in the formulation of the finite-horizon linear quadratic optimal control problem associated with the spectral matrix $\Psi(z)$. See [27] for an account of this connection for the finite-horizon linear quadratic regulator (where the RDE evolves in *reverse* time). □

The Doubling Algorithm.

The above lemma gives rise to the doubling algorithm as a means of solving the discrete-time algebraic Riccati equation. The key observations made in conjunction with Lemma 6.7.1 are that for all $n \in \{0,1,2,\ldots\}$,

$$\begin{pmatrix} X_{2^{n+1}} \\ Y_{2^{n+1}} \end{pmatrix} = \Gamma^{2^{n+1}} \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} \tag{6.98}$$

and that

$$\Gamma^{2^{n+1}}=\Gamma^{2^n}\Gamma^{2^n}. \tag{6.99}$$

Successive iteration of the formula (6.99) results in an iterative scheme for calculating Γ^{2^n}. Using the equation $\hat{\Phi}_k = Y_kX_k^{-1}$ given in item *2* of Lemma 6.7.1, one can then calculate $\hat{\Phi}_{2^n}$ directly. The main advantage with this approach is that these iterates can be calculated *without* having to obtain the intermediate RDE iterates. Note that for the purposes of determining the limiting behaviour of the difference equation (which is our main objective here), the intermediate iterates are irrelevant anyway.

Reduced Order Formulae for the Doubling Iteration.

Two potential problems can be identified with direct application of the iteration (6.99). The first is that the dimensions of the matrices involved can be quite large viz. $2n \times 2n$, where $F \in \mathbb{R}^{n\times n}$. The second is that Γ will generally have eigenvalues with modulus greater than unity. These will dominate and reduce the numerical accuracy of the calculations associated with invariant subspaces corresponding to eigenvalues with smaller modulus. Due to the symplectic structure of Γ, a reexpression of the doubling algorithm is possible which allows the $2n \times 2n$ multiplication in (6.99) to be reduced to a set of three $n \times n$ iterations and which has been reported (see [7]) to result in good numerical behaviour.

In [7], the reduced order doubling formulae were introduced for the RDE associated with a Kalman filtering problem. In that case, it is known that the stabilizing solution of the RDE Φ satisfies $\Phi \geq 0$. In what follows, we shall in fact assume $\Phi \leq 0$ to establish the reduced order formulae. When using the doubling algorithm to solve the DARE constructed from a continuous time spectral matrix via the bilinear transform, this is no loss of generality; it will become apparent in the next section that any continuous time spectral factorization problem can be transformed into one in which $\Phi = P \leq 0$. It should be noted that this is automatically the case for the continuous time bounded real lemma.

Lemma 6.7.2 *Suppose one is given a discrete-time spectral matrix $\Psi(z)$, with a realization given in (6.81) which satisfies assumptions* **DA.2** *and* **DA.4**. *Suppose $\hat{\Phi}$ is the strong solution of the discrete algebraic Riccati equation (6.82) associated with this realization which also satisfies $\hat{\Phi} \leq 0$. Let $\hat{\Phi}_0 \geq \hat{\Phi}$ be the initial condition of the associated Riccati difference equation (6.88). Then the following statements hold:*

1. *For each iteration n of the doubling algorithm $n \in \{0, 1, 2, \ldots\}$, Γ^{2^n} is a symplectic matrix and has the structure*

$$\Gamma^{2^n} = \begin{pmatrix} \alpha_n^{-1} & \alpha_n^{-1}\beta_n \\ \gamma_n\alpha_n^{-1} & \alpha_n^T + \gamma_n\alpha_n^{-1}\beta_n \end{pmatrix}, \tag{6.100}$$

with entries defined via the following iterations

$$\begin{aligned} \alpha_{n+1} &= \alpha_n(I + \beta_n\gamma_n)^{-1}\alpha_n, & (6.101)\\ \beta_{n+1} &= \beta_n + \alpha_n(I + \beta_n\gamma_n)^{-1}\beta_n\alpha_n^T, & (6.102)\\ \gamma_{n+1} &= \gamma_n + \alpha_n^T\gamma_n(I + \beta_n\gamma_n)^{-1}\alpha_n, & (6.103) \end{aligned}$$

which have the initial conditions:
$\alpha_0 = \hat{F} - \hat{G}\hat{U}^{-1}\hat{S}^T$, $\beta_0 = \hat{G}\hat{U}^{-1}\hat{G}^T$ *and* $\gamma_0 = \hat{V} - \hat{S}\hat{U}^{-1}\hat{S}^T$.

2. *The iterates $\hat{\Phi}_{2^n}$ of the RDE (6.88) with initial condition $\hat{\Phi}_0$ can be constructed from the doubling iterates as follows:*

$$\hat{\Phi}_{2^n} = \left(\gamma_n\alpha_n^{-1} + (\alpha_n^T + \gamma_n\alpha_n^{-1}\beta_n)\hat{\Phi}_0\right)\left(\alpha_n^{-1} + \alpha_n^{-1}\beta_n\hat{\Phi}_0\right)^{-1}. \tag{6.104}$$

Proof:

1. Note that Γ^{2^n} is symplectic since any power of a symplectic matrix also has the symplectic property (see [7]). In the discussion on doubling algorithms in section 6.7 of

[7], the observation is made that when Γ^{2^n} is symplectic with an invertible $(1,1)$ block, it can, without loss of generality, be expressed in the form shown in (6.100). We now show that, for the class of realizations of spectral matrices we consider, the $(1,1)$ entry of Γ^{2^n} must always be nonsingular. For any such n, let Γ^{2^n} have partitioning conformal with that of Γ:

$$\Gamma^{2^n} = \begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix}. \tag{6.105}$$

(Note that we have *not* assumed at this stage that A_n is nonsingular.) Recall from item *1* of Lemma 6.7.1 that with $X_0 = I$ and $Y_0 = \hat{\Phi}_0$, $X_{2^n} = A_n + B_n Y_0$ is invertible for all $n \geq 1$. Given the hypothesis that $\hat{\Phi} \leq 0$, then one can take $Y_0 = \hat{\Phi}_0 = 0 \geq \hat{\Phi}$ and therefore deduce immediately that $X_n = A_n$ is nonsingular for all n. Thus for any $n \geq 0$, there exists an invertible matrix $\alpha_n = A_n^{-1}$ and, due to the known symplectic structure of Γ^{2^n}, there also exist matrices β_n and γ_n such that Γ^{2^n} is given by (6.100). The proof of (6.101), (6.102) and (6.103) is an immediate consequence of the observation that $\Gamma^{2^{n+1}} = \Gamma^{2^n}\Gamma^{2^n}$.

2. Note first that

$$\begin{pmatrix} X_{2^n} \\ Y_{2^n} \end{pmatrix} = \Gamma^{2^n} \begin{pmatrix} X_0 \\ Y_0 \end{pmatrix}. \tag{6.106}$$

Equation (6.104) follows from the structure of Γ^{2^n} revealed in item *1* of this lemma and the formula for $\hat{\Phi}_k$ given in item *2* of Lemma 6.7.1. □

Remarks:

a) The proof of the above lemma depends fairly strongly on the assumption that the strong solution of the associated DARE satisfies $\hat{\Phi} \leq 0$. It should be noted that whilst this is a *sufficient* condition for the reduced order doubling iteration to be applicable, it is not a necessary condition. This is illustrated by the well known case of Kalman filtering where generally $\hat{U} > 0$, $\hat{V} \geq 0$ and $\hat{\Phi} \geq 0$.

b) Whilst the underlying dynamical system associated with Γ is linear, the reduced order doubling iterations (6.101), (6.102) and (6.103) are clearly nonlinear. Although an improvement over the direct doubling approach in (6.99), the numerical behaviour of these nonlinearities may impose a further limitation on the achievable accuracy. A closer study of these formulae might prove to be important in a study of the numerical properties of the algorithm. □

6.8 Algorithms for Continuous and Discrete Time Spectral Factorization.

The results presented in section 6.6 on Riccati difference equation convergence and those presented in section 6.7 on the doubling algorithm together make possible an algorithm for solving the algebraic Riccati equation associated with discrete time spectral matrices. Given a continuous time spectral matrix, it was shown in section 6.5 how to construct a family of discrete time problems. A solution to any one of these discrete time problems immediately furnishes a solution to the continuous time problem. This observation, together with the discrete time spectral factorization algorithm results in an algorithm for continuous time spectral factorization. In this section, we give a summary of both the discrete time spectral factorization algorithm and the resulting continuous time algorithm.

In what follows, we assume stabilizability of the underlying linear dynamics and non-singularity (almost everywhere) of the spectral matrix. These assumptions have been made specifically with application to the bounded real lemma in mind. A number of other assumptions on the spectral matrix realization are necessary for direct application of the algorithm. It is shown how these can be circumvented should the spectral matrix realization not satisfy them. The focus on the role of imaginary axis and unit circle invariant zeros is critical to the present development. Results on the convergence rate of the RDE as established in chapter 7 (which were summarized in section 6.6) are used to establish convergence rates for the proposed algorithms.

6.8.1 Discrete Time Spectral Factorization Algorithm.

Suppose one is give a realization of a nonsingular discrete time spectral matrix

$$\Psi(z) = \hat{U} + \hat{G}^T(z^{-1}I - \hat{F}^T)^{-1}\hat{S} + \hat{S}^T(zI - \hat{F})^{-1}\hat{G} + \hat{G}^T(z^{-1}I - \hat{F}^T)^{-1}\hat{V}(zI - \hat{F})^{-1}\hat{G}, \tag{6.107}$$

with the associated discrete time algebraic Riccati equation

$$\hat{\Phi} = \hat{F}^T\hat{\Phi}\hat{F} + \hat{V} - (\hat{F}^T\hat{\Phi}\hat{G} + \hat{S})(\hat{U} + \hat{G}^T\hat{\Phi}\hat{G})^{-1}(\hat{F}^T\hat{\Phi}\hat{G} + \hat{S})^T. \tag{6.108}$$

The algorithm for solving (6.108) (which follows) rests on a number of results established earlier in this chapter. We first summarize the conditions under which all of these results are guaranteed to simultaneously hold:

Assumptions on the Discrete Spectral Matrix Realization.

1. **DA.1** $(\hat{F}, \hat{G})$ is stabilizable.
2. **DA.2** $\hat{U}$ is nonsingular.
3. **DA.4** $\hat{F} - \hat{G}\hat{U}^{-1}\hat{S}^T$ is nonsingular.
4. A strong solution $\hat{\Phi}$ of the associated DARE exists.
5. The strong solution satisfies $\hat{\Phi} \leq 0$.

An Algorithm for Solving the Discrete Time ARE.

Step 1

Set $n = 0$. Initialize the iterates α_0, β_0 and γ_0 as given in item *1* of Lemma 6.7.2.

Step 2

Calculate α_{n+1},β_{n+1} and γ_{n+1} from α_n,β_n and γ_n using the reduced-order doubling formulae (6.101), (6.102) and (6.103).

Step 3

The RDE iterate $\hat{\Phi}_{2^n}$ resulting from the initial condition $\hat{\Phi}_0 = 0$ can be calculated as follows:

$$\hat{\Phi}_{2^n} = \gamma_n. \tag{6.109}$$

Step 4

For a pre-specified tolerance $\epsilon > 0$ in a chosen matrix norm $\|\cdot\|$, determine whether $\|\hat{\Phi}_{2^n} - \hat{\Phi}_{2^{n-1}}\| < \epsilon$?

If YES, then HALT.
If NO, then increment n and go back to **Step 2.**

Remarks:

a) The proofs given in earlier sections leading to the above algorithm guarantee that iterates will converge (in exact arithmetic) under the conditions 1–5 given above. This algorithm might also be directly applicable to many problems for which some of the conditions 1–5 are relaxed, however this possibility is not investigated here.
b) The particular initial condition $\hat{\Phi}_0 = 0$ is used in the above algorithm. This could be replaced by any initial condition $\hat{\Phi}_0 \geq \hat{\Phi}$, where $\hat{\Phi}$ is the strong solution of the DARE. In this case, the general formula (6.104) would need to be applied to obtain $\hat{\Phi}_{2^n}$. □

6.8.2 Continuous Time Spectral Factorization Algorithm.

The continuous time algorithm follows immediately from the construction of the family of related discrete time problems presented in section 6.5, together with the discrete time algorithm presented in the previous subsection.

Suppose one is give a realization of a nonsingular continuous time spectral matrix

$$\Phi(s) = U+B^T(-sI-A^T)^{-1}V(sI-A)^{-1}B+B^T(-sI-A^T)^{-1}S+S^T(sI-A)^{-1}B. \tag{6.110}$$

with the associated continuous time algebraic Riccati equation

$$PA + A^TP + V - (PB + S)U^{-1}(PB + S)^T = 0. \tag{6.111}$$

We now list the conditions introduced in this chapter which together guarantee the applicability of the algorithm presented below.

Assumptions on the Spectral Matrix.

1. **CA.1** (A, B) is stabilizable.

2. **CA.2** U is positive definite.

3. The unique strong solution P of the ARE (6.111) satisfies $P \leq 0$.
   Recall from Lemma 6.1.1 that a strong solution P of the ARE (6.111) is guaranteed to exist since the spectral property $\Phi(j\omega) \geq 0$ holds in addition to the assumptions **CA.1** and **CA.2**.

Remarks:

a) Condition *3* is at first sight quite strong and clearly restricts the class of spectral matrices to which the algorithm is *directly* applicable. This limitation arises due to the restricted conditions under which the discrete time spectral factorization algorithm is guaranteed to work.

b) We now show that condition *3* holds for an important subclass of spectral matrices. Recall that the main motivation in this thesis for solving the spectral factorization Riccati equation is the bounded real lemma. Suppose one is given a transfer function matrix $M(s) = C(sI - A)^{-1}B$ with A stable. To test whether $M(s)$ is bounded real, one can consider the spectral matrix $\Phi(s) = I - M^T(-s)M(s)$. It is easily checked that the associated ARE has the form $PA + A^TP - (PBB^TP + C^TC) = 0$. Since A is stable, it follows from standard results on Lyapunov equations that $P \leq 0$ (see Lemma 0.0.1 in the summary of **Notation, Definitions and Fundamental Results** at the beginning of this thesis).

c) It should be noted that the algorithm described below may well still work in many circumstances where the condition $P \leq 0$ does not hold.

d) It is explained in Appendix G.2 and a summary is now given of how the limitation that $\hat{\Phi} \leq 0$ can be circumvented, this making the algorithm applicable to *any* continuous time spectral matrix which satisfies **CA.1** and **CA.2**.

- Firstly, by the choice of any state feedback law K which ensures that $A_K = A - BK$ is stable, one can construct a related spectral matrix

$$\Phi_K(s) = U + B^T(-sI - A_K^T)^{-1}V_K(sI - A_K)^{-1}B + \tag{6.112}$$
$$B^T(-sI - A_K^T)^{-1}S_K + S_K^T(sI - A_K)^{-1}B, \tag{6.113}$$

  where state-space formulae for A_K, V_K and S_K are given in (G.12) in Appendix G.2. Importantly, the algebraic Riccati equation associated with the above realization of this new spectral matrix is *identical* to that associated with the original realization of $\Phi(s)$.

- Next a different realization of the spectral matrix $\Phi_K(s)$ is constructed using Lemma 6.1.1 of this chapter; with $'P'$ in the statement of that Lemma (i.e. *not* the strong solution P of the ARE) chosen to be the unique (symmetric) solution M of the Lyapunov equation $MA_K + A_K^TM = -V_K$ (the existence of a solution to this equation is guaranteed by the stability of A_K; see Lemma 0.0.1 in the summary of **Notation, Definitions and Fundamental Results** at the beginning of this thesis). The realization which results can be constructed by applying the formula (G.14) in Appendix G.2 to the realization of $\Phi_K(s)$ given in (6.113):

$$\Phi_K(s) = U + B^T(-sI - A_K^T)^{-1}(S_K + MB) + (S_K + MB)^T(sI - A_K)^{-1}B. \quad (6.114)$$

It is shown in the proof of Lemma G.2.2 in Appendix G.2 that the ARE associated with this new realization of $\Phi_K(s)$ will *always* have a strong solution Π_2 which satisfies $\Pi_2 \leq 0$. (Note that $\Phi_K(s)$ is found without knowing Π_2.)

- It is now quite straightforward to check that the that the new realization of Φ_K in (6.114) satisfies each of the conditions $1, \ldots, 3$ listed above for continuous time spectral matrices. This makes it possible to apply the algorithm described below to find the solution Π_2 of the associated ARE.

- It is also shown in Appendix G.2 that, having found Π_2, one can construct the solution P of the original ARE (6.111) as $P = \Pi_2 + M$.

□

An Algorithm for Solving the Continuous Time ARE.

Step 1

With reference to Lemma 6.5.2, construct a realization of a discrete time spectral matrix $\Psi_\alpha(z)$ from the realization of $\Phi(s)$ in (6.111) as follows:

$$\Psi_\alpha(z) = \begin{pmatrix} \hat{G}_\alpha^T(z^{-1}I - \hat{F}_\alpha^T)^{-1} & I \end{pmatrix} \begin{pmatrix} \hat{V}_\alpha & \hat{S}_\alpha \\ \hat{S}_\alpha^T & \hat{U}_\alpha \end{pmatrix} \begin{pmatrix} (zI - \hat{F}_\alpha)^{-1}\hat{G}_\alpha \\ I \end{pmatrix}, \quad (6.115)$$

where formulae for each of the above matrices are given in Lemma 6.5.2. Here $\alpha > 0$ is a real number chosen such that it is not an eigenvalue of A and such that $\hat{U}_\alpha$ is nonsingular.

Step 2

Now consider application of the algorithm for solving discrete time AREs (described in subsection 6.8.1) to the spectral matrix $\Psi(z) = \Psi_\alpha(z)$, with realization given in (6.115). We now check that each of the conditions $1, \ldots, 5$ required for application of the DARE algorithm in section 6.8.1 hold:

1. It is shown in Lemma 6.5.2 that **CA.1** implies $(\hat{F}_\alpha, \hat{G}_\alpha)$ is stabilizable.

2. It is also shown in Lemma 6.5.2 that $\hat{U}_\alpha$ is generically nonsingular as a function of α.

3. The quantity $\hat{F}_\alpha - \hat{G}_\alpha \hat{U}_\alpha^{-1}\hat{S}_\alpha^T$ is also generically nonsingular as a function of α. This follows from the following facts; in the limit as $\alpha \to \infty$, $\hat{F}_\alpha \to -I$, $\hat{G}_\alpha \to 0$, $\hat{U}_\alpha \to U$ (since $D_\alpha \to 0$) and $\hat{S}_\alpha \to 0$ (since $C_\alpha \to 0$). Thus $\lim_{\alpha\to\infty} \hat{F}_\alpha - \hat{G}_\alpha \hat{U}_\alpha^{-1}\hat{S}_\alpha^T = -I$, which establishes the result.

4. Recall that if the spectral property $\Phi(j\omega) \geq 0$ holds then we can be assured of the existence of a strong solution P of the continuous ARE. It follows from Lemma 6.5.1 that $\hat{\Phi} = \hat{\Phi}_\alpha = P$ is a strong solution of the DARE associated with $\Psi_\alpha(z)$.

5. This follows since it has been assumed for the continuous time problem that $P \leq 0$.

Step 3 Apply the algorithm for solving the discrete time ARE directly to the above realization of $\Psi_\alpha(z)$ to find an approximation to $\hat{\Phi}_\alpha = P$ with specified tolerance ϵ.

Remark: The above actually describes a *family* of algorithms which is parametrized by the variable α. In general, the convergence properties of the RDE will vary, according to the value of the parameter α. Some analysis and findings from numerical simulation studies appear in the context of a study of the linear quadratic control problem in the paper [37]. □

6.8.3 Convergence Properties of the Spectral Factorization Algorithms.

Given exact arithmetic, if a strong solution to the continuous/discrete Riccati equation exists, the algorithms given in the previous two subsections should always halt due to the established guaranteed convergence results for Riccati difference equations. The result will be a matrix $\hat{P}$ which is close to the actual strong solution P. In this subsection, we summarize the rate of convergence of the algorithm in exact arithmetic. The question of convergence behaviour in finite arithmetic and the achievable numerical accuracy of solutions is not considered here. However, the rates presented below would have direct relevance to an investigation of finite wordlength effects.

Convergence Rates for the Spectral Factorization Algorithm.

We state the convergence results for the discrete time algorithm only since those for the continuous time case can be readily inferred from these.

- Suppose the realization of $\Psi(z)$ has an invariant zero on the unit circle; then the following convergence results hold for the algorithm proposed in subsection 6.8.1.

  There exist constants $\hat{\kappa}_1, \hat{\kappa}_2$ (depending on the realization of $\Psi(\cdot)$ and on $\hat{\Phi}_0$) with $\hat{\kappa}_1 > \hat{\kappa}_2 > 0$ such that:

  Worst Case For all $\hat{\epsilon} > 0$, there exists a $n_{\hat{\epsilon}}$ such that when $n \geq n_{\hat{\epsilon}}$

$$\lambda_{\max}(\hat{\Phi}_{2^n} - \hat{\Phi}) \leq \frac{\kappa_1 + \hat{\epsilon}}{2^n}. \tag{6.116}$$

Best Case If the initial condition also satisfies $\hat{\Phi}_0 > \hat{\Phi}$, then for all $\hat{\eta} \in (0, \hat{\kappa}_2)$, there exists a $n_{\hat{\eta}}$ such that when $n \geq n_{\hat{\eta}}$,

$$\lambda_{\max}(\hat{\Phi}_{2^n} - \hat{\Phi}) \geq \frac{\hat{\kappa}_2 - \hat{\eta}}{2^n}. \tag{6.117}$$

- If $\Psi(z)$ has no unit circle invariant zeros, then there exist constants $A > 0$ and $1 > c > 0$ such that

$$\lambda_{\max}(\Phi_{2^n} - \Phi) \leq Ac^{2^n} \tag{6.118}$$

The convergence rate of the algorithm not only describes the rate at which large deviations (e.g. the initial deviation Δ_0) decay, but also the ability of the algorithm to *attenuate numerical errors*. It should be noted that the slower convergence rate for the unit circle case will decrease the achievable numerical accuracy of the algorithm in finite arithmetic. This is in accord with the observations in section 6.3 of this chapter. There it was shown that the presence of imaginary axis invariant zeros in the continuous time case leads to reduced numerical accuracy due to ill conditioning of the problem.

6.9 Conclusions about the Iterative Algorithm for Spectral Factorization.

In this chapter we have reviewed the task of construction of strong solutions of the algebraic Riccati equations which arise in continuous and discrete spectral factorization. Algorithms for solving the ARE which are implemented in standard software packages are based on calculation of a basis for the stable invariant subspace of a Hamiltonian matrix which can be constructed using the state space data for the spectral matrix. These algorithms fail when imaginary axis invariant zeros are present due to the fact that they ignore the Hamiltonian structure in the invariant subspace calculation. More advanced techniques based on invariant subspace calculations which respect this structure have been developed in the literature but are more complex.

One approach to solving the ARE which is comparatively simple is based on the matrix sign iteration. This involves no explicit invariant subspace calculation but repeated application of a simple and direct iterative formula to the original state space data. However, it is well known that the classical matrix sign iteration is not guaranteed to work when applied to the Hamiltonian matrix associated with a continuous-time spectral matrix which has invariant zeros on the imaginary axis.

The algorithm proposed here is, like the matrix sign algorithm, based on a straightforward iterative formula. A discrete time spectral matrix can be constructed from a continuous time spectral matrix using a bilinear transformation. It is shown that the strong solution of the continuous time ARE is also a strong solution of the discrete ARE associated with the discrete time problem. It is shown that strong solutions of the discrete-time Riccati equation can be found via the convergent sequence of iterates of a

Riccati difference equation associated with the discrete time spectral matrix. Should the spectral matrix have imaginary axis invariant zeros, then the convergence rate is $\mathcal{O}(\frac{1}{k})$ in general which is comparatively slow. A doubling algorithm for RDEs in discrete time spectral factorization allows the convergence rate to be substantially increased. The resulting algorithm has a convergence rate of $\mathcal{O}(\frac{1}{2^n})$ in exact arithmetic. This convergence rate should provide a basis for future work assessing the performance of the algorithm when implemented using finite arithmetic.

Chapter 7

Convergence Rates of Riccati Difference Equations for Discrete Time Spectral Factorization with Unit Circle Invariant Zeros.

Summary.

Given a state-space realization of a generically nonsingular discrete time spectral matrix, it is well known that minimum phase spectral factors can be constructed from the strong solution of an associated discrete algebraic Riccati equation (DARE), (see the review in section 6.4). Closely associated with the algebraic equation is a Riccati difference equation (RDE). The convergence behaviour of the iterates of this equation towards the strong solution of the DARE is the subject of the present chapter. Fairly mild conditions are adopted concerning the realization of the spectral matrix and the initial condition of the difference equation. Convergence results for RDEs which arise in a number of special cases of the discrete time spectral factorization problem are available in the literature and these are summarized in the first section of this chapter. Few results are available, however which discuss in detail convergence of the Riccati difference equation when the realization of the discrete time spectral matrix has unit circle invariant zeros. The main objective of this chapter is to derive RDE convergence *rates* for this case.

It should be noted that the question of RDE convergence arose in section 6.6 of Chapter 6. In Chapter 6, it was shown how a family of discrete spectral matrices can be constructed from a continuous time spectral matrix and how the continuous time spectral factorization problem can be solved via the discrete time problem. Should the continuous time spectral matrix have imaginary axis invariant zeros, then each member of the resulting family of discrete time spectral matrices will have unit circle invariant zeros. Discrete time spectral matrices with unit circle invariant zeros are also important for a number of other reasons. Discrete time spectral matrices that have unit circle transmission zeros can arise in the discrete bounded-real and positive-real lemmas. In addition, spectral matrices with unit circle invariant (but not transmission) zeros arise

when considering Kalman filtering for systems which have state dynamics with unit circle modes which are not corrupted by process noise.

The key observations made in this chapter concern the fine structure of the Riccati difference equation iterates when unit circle invariant zeros are present. From these results it is shown that there exists a worst case convergence rate of $\mathcal{O}(\frac{1}{k})$, where k ranges over the nonnegative integers. It is also shown that generically, the convergence will not be faster than this. It should be noted that the nonnegative definiteness of the spectral matrix on the unit circle (spectral property) plays an important role in establishing these convergence rate results. Knowledge of the best and worst case convergence rates is of importance when using the Riccati Difference Equation as a means of finding the strong solution of the associated algebraic equation, as in section 6.6 of Chapter 6. The results established in this chapter can also be used to describe the convergence rate of state covariance matrices to their steady state values in Kalman filtering for systems with noise-free unit-circle modes.

7.1 Introduction.

The first step in this section is to state the main problem addressed in this chapter which concerns Riccati difference equation convergence for discrete time spectral factorization. Of central interest are spectral matrices with unit circle zeros. Next, and with reference to the literature, a discussion is included as to how these unit circle zeros may arise in different spectral factorization problems. Finally, a survey of the literature relevant to RDE convergence in such cases is given. Particular attention is given to evidence in the literature regarding convergence rates in the case of unit circle invariant zeros.

7.1.1 Problem Statement.

Recall the class of realizations of nonsingular discrete time spectral matrices $\Psi(z)$ which was introduced in section 6.4.1 of chapter 6:

$$\Psi(z) = \begin{pmatrix} \hat{G}^T(z^{-1}I - \hat{F}^T)^{-1} & I \end{pmatrix} \begin{pmatrix} \hat{V} & \hat{S} \\ \hat{S}^T & \hat{U} \end{pmatrix} \begin{pmatrix} (zI - \hat{F})^{-1}\hat{G} \\ I \end{pmatrix}. \tag{7.1}$$

where each constant matrix is real, $\hat{V} = \hat{V}^T$ and $\hat{U} = \hat{U}^T$ and where we make the following assumptions:

DA.1 $(\hat{F}, \hat{G})$ is stabilizable.

DA.2 $\hat{U}$ is nonsingular.

It should be emphasized that it has not been assumed tha the matrix $\begin{pmatrix} \hat{V} & \hat{S} \\ \hat{S}^T & \hat{V} \end{pmatrix}$ is sign definite. Now consider the *discrete algebraic Riccati equation (DARE)* associated with the above realization of the spectral matrix:

$$\hat{\Phi} = \hat{F}^T\hat{\Phi}\hat{F} + \hat{V} - (\hat{F}^T\hat{\Phi}\hat{G} + \hat{S})(\hat{U} + \hat{G}^T\hat{\Phi}\hat{G})^{-1}(\hat{F}^T\hat{\Phi}\hat{G} + \hat{S})^T. \tag{7.2}$$

The role of the DARE in solving the discrete spectral factorization problem was discussed in section 6.4.3.

Assumed Existence of a Strong DARE Solution.

We shall assume throughout this chapter the existence of a *strong* solution of (7.2). Recall that a solution of the DARE is by definition strong if the matrix $\hat{F} - \hat{G}N^{-1}(\hat{G}^T\hat{\Phi}\hat{F} + \hat{S}^T)$ has all eigenvalues in the closed unit circle, where $N = \hat{U} + \hat{G}^T\hat{\Phi}\hat{G}$. Recall also that since $\Psi(z)$ is a nonsingular spectral matrix, it follows that $N > 0$.

Questions about Riccati Difference Equation Convergence.

The main aim of the present chapter is to develop an understanding of the behaviour of the related *Riccati difference equation (RDE)* with initial condition $\hat{\Phi}_0$:

$$\hat{\Phi}_{k+1} = \hat{F}^T\hat{\Phi}_k\hat{F} + \hat{V} - (\hat{F}^T\hat{\Phi}_k\hat{G} + \hat{S})(\hat{U} + \hat{G}^T\hat{\Phi}_k\hat{G})^{-1}(\hat{F}^T\hat{\Phi}_k\hat{G} + \hat{S})^T. \tag{7.3}$$

With reference to this equation, we shall be concerned with the following questions:

- Under what conditions on the realization of the spectral matrix and the initial condition $\hat{\Phi}_0$ will iterates of the RDE converge to the strong solution of the DARE?
- Does there exist an upper bound on the *rate* at which the iterates converge?
- Does there exist a lower bound on the *rate* at which the iterates converge?

In describing bounds on the rate of convergence, we would hope to establish the existence of functions $\bar{e}(k, \hat{\Phi}_0)$ and $\underline{e}(k, \hat{\Phi}_0)$ which are nonnegative, converge to zero at known rates and satisfy the following inequalities:

$$\underline{e}(k, \hat{\Phi}_0) \leq |\lambda_{\max}(\hat{\Phi}_k - \hat{\Phi})| \leq \bar{e}(k, \hat{\Phi}_0). \tag{7.4}$$

Realizations of spectral matrices which have unit circle invariant zeros figure prominently in the convergence analysis. In fact, it will become apparent that unit circle invariant zeros pose a fundamental limitation on the rate at which RDE iterates can converge to the strong solution of the ARE. On the other hand, we demonstrate that an upper bound on the convergence rate can also be guaranteed when unit circle invariant zeros are present.

Henceforth, for brevity, we shall refer to the invariant realizations of a particular realization of a spectral matrix $\Psi(z)$ simply as the invariant zeros of $\Psi(z)$, provided it is clear from the context which realization is meant.

Remarks:

a) In the derivation of convergence rates of the above form, exact arithmetic is assumed.

b) Convergence rates of the above form have immediate benefit in describing the rate at which iterates converge to a *neighbourhood* of the strong solution.
c) Once the RDE iterates get close to the DARE solution, the convergence rate bounds will be of likely benefit in describing the numerical accuracy of the iteration. However this issue is not investigated here. □

7.1.2 Unit Circle Invariant Zeros in Discrete Time Spectral Factorization.

Before proceeding, we shall review the role of unit circle invariant zeros in discrete time spectral factorization problems. In the previous chapter, a description is given as to how realizations of discrete time spectral matrices which have unit circle invariant zeros arise in consideration of continuous time spectral factorization problems. However, unit circle invariant zeros can arise in their own right in a number of inherently discrete problems.

Linear Quadratic Regulator and Kalman Filtering.

In $\mathcal{H}_2$ linear-quadratic optimal control and Kalman filtering, spectral matrices arise which are special cases of the class (7.1) where it is generally assumed that $\hat{S} = 0$, $\hat{U} > 0$ and $\hat{V} \geq 0$. In such cases the spectral matrix is guaranteed to be positive definite on the unit circle and clearly the possibility that the spectral matrix has unit circle transmission zeros is precluded due to the sign definiteness of $\hat{U}$ and $\hat{V}$. In much of the literature, unit circle invariant zeros are precluded altogether by assuming detectability or observability of the pair $(\hat{V}^{\frac{1}{2}}, \hat{F})$ (see for example [37]).

Note, however, that unit circle *decoupling* zeros can appear if the realization of the spectral matrix is nonminimal: for example if $(\hat{V}^{\frac{1}{2}}, \hat{F})$ has unobservable unit-circle modes (see e.g. [3, 15, 22]), then these become decoupling zeros of the spectral matrix realization. It should be noted that in each of [3, 15, 22], Kalman filtering problems are treated in which filtered estimates are sought of the states of a system $x_{k+1} = Ax_k + w_k$ from outputs $y_k = Cx_k + v_k$. In terms of the notation adopted in the present chapter, $A = \hat{F}^T$, $C = \hat{G}^T$, $\mathcal{E}\{v_k v_k^T\} = \hat{U}$ and $\mathcal{E}\{w_k w_k^T\} = \hat{V}$. Thus if $(\hat{V}^{\frac{1}{2}}, \hat{F})$ has unobservable modes, it follows that $(A, \hat{V}^{\frac{1}{2}})$ has uncontrollable modes (recall that $\hat{V}^{\frac{1}{2}}$ is symmetric). Since $\hat{V}$ is interpreted as the process noise covariance matrix, this says that some modes of the underlying linear system are uncorrupted by process noise. In [3, 15, 22], there are modes of this type on the unit circle.

The Discrete Positive Real Lemma.

A square transfer function matrix $G(z) = D + C(zI - A)^{-1}B$ is discrete positive real (see e.g. [36, 4]) if outside the unit circle, both of the following are true: $G(z)$ is analytic and $G^T(z^*) + G(z) \geq 0$. Associated with $G(z)$, one can define a spectral matrix $\Psi(z) = G^T(z^{-1}) + G(z)$. In terms of the present notation for the state-space realization of the spectral matrices, when considering such problems, it is easy to check that the matrix $\hat{V} = 0$. Therefore, if the matrix A has unit circle eigenvalues, they automatically become invariant (decoupling) zeros of $\Psi(z)$. It may also be the case that the spectral matrix has unit circle transmission (and hence invariant) zeros.

The Discrete Bounded Real Lemma.

Unit circle transmission zeros can also arise in the discrete time version of the bounded-real lemma (see e.g. [24]) which is relevant in the discrete $\mathcal{H}_\infty$ control problem (see also [105]). Recall that a discrete-time transfer function matrix $L(z)$ is called *bounded real* if all poles of $L(z)$ are inside the unit circle and $\|L\|_\infty \leq 1$. Consider the spectral matrix $\Psi(z)$ defined by $\Psi(z) = I - L^T(z^{-1})L(z)$. Observe that with the state-space realization $L(z) = H_L(zI - F_L)^{-1}G_L$, $\Psi(z)$ is of the standard form given in (7.1), with $\hat{U} = I$, $\hat{F} = F_L$, $\hat{G} = G_L$ and $\hat{V} = -H_L^T H_L$. Should $\sigma_{\max}(L(e^{j\theta^*})) = 1$ for some θ^*, then $\Psi(e^{j\theta^*})$ loses rank at that point; i.e. $\Psi(z)$ has a transmission zero at $e^{j\theta^*}$. In fact the above example is a discrete time version of the continuous time bounded real problem which was described in the introduction of this thesis in the context of multiple objective robust control. It is likely (though not investigated here) that the boundary of the bounded real constraint will play a role in discrete time multiple objective robust control analogous to the role it plays in the continuous time case.

7.1.3 A Summary of known RDE Convergence Results.

We now review the literature concerning the convergence of RDEs, paying particular attention to the role of unit circle invariant zeros and to the available convergence *rate* results. We also compare the assumptions on the state space realizations with those adopted here.

The Discrete Bounded Real Lemma.

To the author's knowledge, the possibility of employing the associated RDE to solve this problem and the related convergence questions have not been studied in the literature. The results established in this chapter can be used to address this problem for a discrete bounded real system $L(z) = H_L(zI - F_L)^{-1}G_L$, provided (F_L, G_L) is stabilizable.

The Discrete Positive Real Lemma.

A Riccati difference equation appears in [4], where discrete time spectral factorization is posed in terms of a finite horizon optimal control problem. Under the assumption that $(\hat{F}, \hat{G})$ is controllable, it is shown in that paper that RDE iterates converge and convergence rates are stated - an exponential rate when no unit circle invariant zeros are present, and a $\frac{1}{k}$ rate when they are present. Whilst steps towards a proof of these results are contained in [4], a full proof is not presented. It is also commented in [4] that $\frac{1}{k}$ rates have been observed in numerical studies. In the present chapter, this convergence behaviour is investigated in detail and the assumption that $(\hat{F}, \hat{G})$ is controllable is relaxed to stabilizable.

In connection with the *continuous-time* positive real lemma in [8], it should be noted that a $\frac{1}{t}$ rate has been established for convergence of a Riccati *differential* equation to the strong solution of the associated algebraic Riccati equation.

Linear Quadratic Regulator and Kalman Filtering.

The Riccati difference equation (7.3) appears in the context of finite-horizon discrete time Kalman filtering [7] (where k is the time index and $\hat{\Phi}_0$ the covariance of the initial state estimate) and finite-horizon optimal control [5] (where k evolves in reverse-time and $\hat{\Phi}_0$ is a terminal-state weighting matrix).

If one adopts the assumptions that $(\hat{V}^{\frac{1}{2}}, \hat{F})$ is detectable and $\hat{\Phi}_0 \geq 0$, iterates of (7.3) are known to converge to the strong solution of the algebraic Riccati equation (see [5] and [9]). It should be noted that a somewhat *stronger* assumption on the initial condition is adopted in the present work: $\hat{\Phi}_0 \geq \hat{\Phi}$, where $\hat{\Phi}$ is the strong solution of the DARE. This result was strengthened in [15] where it is shown that convergence is guaranteed if $(\hat{V}^{\frac{1}{2}}, \hat{F})$ has no unobservable modes on the unit circle and $\hat{\Phi}_0 > 0$. Note that in [5], [9] and [15], the possibility that the spectral matrix has invariant zeros on the unit circle is precluded and the strong solution is actually a stabilizing solution. In this case, strong solutions of the ARE (7.2) also satisfy $\hat{\Phi} \geq 0$ (which is *not* true in general of the broader class of spectral matrices considered here.) An exponential convergence rate for the RDE in linear filtering and control problems under the above assumptions is derived in [13].

Whilst unit circle *transmission* zeros are precluded in the standard filtering and LQ regulator problems, recall that *invariant* zeros may exist if the realization of the spectral matrix is nonminimal: for example if $(\hat{V}^{\frac{1}{2}}, \hat{F})$ has unobservable unit-circle modes, these become decoupling zeros of the spectral matrix. One of the first observations of the behaviour of the RDE in such circumstances is made in [3]. In that paper, a scalar example is considered with an identity state mapping, no process noise and with observations corrupted with Gaussian white noise. For this example, the iterates of the Kalman Filter RDE are shown to converge at a $\frac{1}{k}$ rate. A similar example can be found in [15].

More recently, the possibility of unit circle invariant zeros arising due to nonminimality of the realization of the spectral matrix has been discussed in [15] and [22] where Kalman filtering for so-called *nonstabilizable* systems is discussed. When expressed in terms of the notation adopted here, in these papers the possibility is considered that $(\hat{V}^{\frac{1}{2}}, \hat{F})$ has unobservable unit circle modes. Under the additional assumption that $\hat{\Phi}_0 \geq \hat{\Phi}$, convergence of the RDE iterates to the strong solution $\hat{\Phi}$ is stated in Theorem 4.2 of [22]. This generalizes the convergence result in Theorem 4.3 of [15] where it is assumed that (F, G) is controllable and either $\hat{\Phi}_0 > \hat{\Phi}$ or $\hat{\Phi}_0 = \hat{\Phi}$. In [22], a finite horizon scalar LQ control example is presented with no control weighting, except on the terminal state. An explicit formula for RDE iterates is derived there which converges at a $\frac{1}{k}$ rate.

It should be noted that convergence of the Riccati *differential* equation for the *continuous-time* LQ regulator problem has been studied in [14]. There, a stabilizability condition is adopted on the state space data and the possibility of uncontrollable *imaginary axis* modes in the realization of the spectral matrix is allowed. It is noted in [14] that a $\frac{1}{t}$ convergence rate is possible in such circumstances.

Some questions regarding RDE Convergence Rates.

The evidence at hand from two scalar filtering examples (in [3] and [22]) and from the discrete positive real lemma (in [4]) indicates that a $\frac{1}{k}$ convergence rate is possible for RDEs associated with spectral matrices with unit circle invariant zeros. Observe, however that there is no proof in the above papers, or elsewhere in the literature it seems, concerning the convergence rate for spectral matrices under the assumptions given in the present chapter. A partial proof of the convergence rate result is given in Appendix III of [4] under the somewhat stronger assumption that $(\hat{F}, \hat{G})$ is controllable. The account given in [4] stops short of a detailed discussion of the mechanism of convergence of the RDE.

Given a realization of a spectral matrix as in (7.1) which satisfies the assumptions **DA.1** and **DA.2**, we consider here the following questions in relation to the associated RDE (7.3):

- Can the convergence rate ever be *worse* than $\frac{1}{k}$?
- Can we ever expect convergence to be *better* than $\frac{1}{k}$?

It will become apparent that the answer to the first question is *no* and that the answer to the second question is *generically no.* Thus faster convergence is possible, but only for non-generic initial conditions $\hat{\Phi}_0$.

7.2 An Equivalent Problem with a Simpler Spectral Matrix Realization.

In this section we show that, without loss of generality, in the remainder of this chapter we can consider a *subclass* of the realizations of nonsingular discrete time spectral matrices considered earlier. Each element $\Psi(z)$ of this subclass can be realized as follows:

$$\Psi(z) = U + G^T(z^{-1}I - F^T)^{-1}V(zI - F)^{-1}G. \tag{7.5}$$

Here, as before, each constant matrix is real, $V = V^T$, $U = U^T$ and the following assumptions hold:

DA.1 (F, G) is stabilizable.

DA.2 U is nonsingular.

The difference here is that it is assumed that the cross-terms are zero; i.e. $S = 0$. The following lemma is the first step in demonstrating that this is no loss of generality.

Lemma 7.2.1 *Let $\Psi(z)$ be a nonsingular spectral matrix with realization given by (7.1). Let K be any state-feedback law (not necessarily stabilizing) for $(\hat{F},\hat{G})$ and define*

$$\begin{aligned}\hat{F}_K &= \hat{F}+\hat{G}K, && (7.6)\\ \hat{S}_K &= \hat{S}+K^T\hat{U}, && (7.7)\\ \hat{V}_K &= \hat{V}+K^T\hat{U}K+\hat{S}K+K^T\hat{S}^T, && (7.8)\end{aligned}$$

then $\Psi(z)$ can be expressed as

$$\Psi(z) = (I-\hat{G}^T(z^{-1}I-\hat{F}^T)^{-1}K^T)\Psi_K(z)(I-K(zI-\hat{F})^{-1}\hat{G}), \tag{7.9}$$

where

$$\Psi_K(z) = \hat{U}+\hat{G}^T(z^{-1}I-\hat{F}_K^T)^{-1}\hat{S}_K+\hat{S}_K^T(zI-\hat{F}_K)^{-1}\hat{G}+\hat{G}^T(z^{-1}I-\hat{F}_K^T)^{-1}\hat{V}_K(zI-\hat{F}_K)^{-1}\hat{G}. \tag{7.10}$$

Moreover, the following hold:

1. *$\Psi(z)$ is a spectral matrix if and only if $\Psi_K(z)$ is a spectral matrix.*
2. *$\Psi(z)$ is nonsingular if and only if $\Psi_K(z)$ is nonsingular.*
3. *$\hat{\Phi}$ is a solution of the algebraic Riccati equation (7.2) associated with the realization of $\Psi(z)$ given in (7.1) if and only if it is a solution of the algebraic Riccati equation associated with the realization of $\Psi_K(z)$ given in (7.10).*
4. *If $(\hat{F},\hat{G})$ is a stabilizable pair if and only if $(\hat{F}_K,\hat{G})$ is a stabilizable pair.*
5. *If $\hat{U}$ is nonsingular, then by choosing $K=-\hat{U}^{-1}\hat{S}^T$, the cross terms are eliminated in the realization of $\Psi_K(z)$ (i.e. $\hat{S}_K=0$).*

Proof: A discussion of how one obtains the family of spectral matrices in (7.10) can be found in [80]. Items *1* and *2* follow immediately from the identity (7.9). Item *3* was identified in [80]. Item *4* is a well known result from linear system theory. Item *5* is easily verified. □

Suppose one is given a spectral matrix $\Psi(z)$ with realization as given in (7.1), defined by the matrices $\{\hat{F},\hat{G},\hat{U},\hat{V},\hat{S}\}$ and which satisfies **DA.1** and **DA.2** . Note that here the cross coupling matrix $\hat{S}$ is nonzero in general. Application of item *5* of the above lemma to this realization results in a *new* spectral matrix $\Psi_K(z)$ with state space realization

$$\Psi(z) = U+G^T(z^{-1}I-F^T)^{-1}V(zI-F)^{-1}G, \tag{7.11}$$

where $U=\hat{U}$ and the other state-space matrices are given by application of the formulae (7.6), (7.7) and (7.8) to the state-space realization of $\Psi(z)$ with $K=-\hat{U}^{-1}\hat{S}^T$;

$$\begin{aligned}F &= \hat{F}-\hat{G}\hat{U}^{-1}\hat{S}^T, && (7.12)\\ G &= \hat{G}, && (7.13)\end{aligned}$$

$$\begin{aligned} S &= 0, & (7.14)\\ V &= \hat{V} - \hat{S}\hat{U}^{-1}\hat{S}^T. & (7.15) \end{aligned}$$

Since $(\hat{F}, \hat{G})$ is a stabilizable pair, it follows from item *4* of the above lemma that (F, G) is also stabilizable.

It follows from item *3* of the above lemma that if $\hat{\Phi}$ is a solution of the DARE (7.2), then $\Phi = \hat{\Phi}$ is also a solution of the following algebraic Riccati equation:

$$\Phi = F^T \left(\Phi - \Phi G(U + G^T \Phi G)^{-1} G^T \Phi\right) F + V. \tag{7.16}$$

Note also that if $\hat{\Phi}$ is a *strong* solution to the DARE (7.2) associated with the realization (7.1) of $\Psi(z)$, then $\Phi = \hat{\Phi}$ is also a strong solution to the above DARE (7.16). This follows from the following (fairly easily established) identity: $F - GN^{-1}G^T\Phi F = \hat{F} - \hat{G}N^{-1}(\hat{G}^T\hat{\Phi}\hat{F} + \hat{S}^T)$, where $N = \hat{U} + \hat{G}^T\hat{\Phi}\hat{G} = U + G^T\Phi G > 0$.

Recall that describing the convergence of the iterates $\{\hat{\Phi}_k\}$ to the strong solution $\hat{\Phi}$ is of central interest in this chapter. In studying the behaviour of the RDE iterates $\{\hat{\Phi}_k\}$, we now show that we can *equivalently* study the iterates $\{\Phi_k\}$ of the following Riccati difference equation, with initial condition $\Phi_0 = \hat{\Phi}_0$:

$$\Phi_{k+1} = F^T \left(\Phi_k - \Phi_k G(U + G^T \Phi_k G)^{-1} G^T \Phi_k\right) F + V. \tag{7.17}$$

Lemma 7.2.2 *Consider iterates $\{\hat{\Phi}_k\}$ and $\{\Phi_k\}$ (for all $k \in \{1, 2, \ldots\}$) of the Riccati difference equations (7.3) and (7.17) respectively. Suppose these RDEs have the same initial condition $\hat{\Phi}_0 = \Phi_0$, then*

$$\hat{\Phi}_k = \Phi_k \quad (\forall\, k \in \{1, 2, \ldots\}).$$

Proof: By substituting the expressions for F and V given in (7.12) and (7.15) in the RDE for Φ_k (7.17), it can be shown using standard matrix manipulations that the right hand side of (7.17) is identical to the right hand side of the RDE for $\hat{\Phi}_k$ (7.3). See [37] for the details of this proof. □

In summary, we can study the RDE iterates $\hat{\Phi}_k$ associated with the realization of a spectral matrix $\Psi(z)$ given in (7.1) by studying the convergence behaviour of the RDE iterates $\{\Phi_k\}$ associated with the realization (7.5) of the related discrete time spectral matrix $\Psi_K(z)$ given in (7.11). There is no loss of generality in doing this since if $\Phi_0 = \hat{\Phi}_0$, then the iterates $\{\Phi_k\}$ arising from the simplified problem (with $S = 0$) are identical to $\hat{\Phi}_k$.

7.3 Comparison Results for RDE Iterates.

We now state a minor extension of a well-known result which describes the way in which Riccati difference equation iterates behave under perturbations to the initial condition Φ_0. Having established this result, we will find it has several applications in the proof of RDE convergence.

Lemma 7.3.1 *Let the sequences $\{\Phi_k^1\}$ and $\{\Phi_k^2\}$ be defined by application of the Riccati Difference Equation (7.17) with initial conditions Φ_0^1 and Φ_0^2 respectively. With the definition $\tilde{\Phi}_k = \Phi_k^2 - \Phi_k^1$,*

1. *The following recursions hold for all $k \geq 0$:*

$$\tilde{\Phi}_{k+1} = (\tilde{F}_k^1)^T \tilde{\Phi}_k \tilde{F}_k^1 - (\tilde{F}_k^1)^T \tilde{\Phi}_k G(G^T \tilde{\Phi}_k G + G^T \Phi_k^1 G + U)^{-1} G^T \tilde{\Phi}_k \tilde{F}_k^1 \tag{7.18}$$

$$\tilde{\Phi}_{k+1} = (\tilde{F}_k^2)^T \tilde{\Phi}_k \tilde{F}_k^2 + (\tilde{F}_k^2)^T \tilde{\Phi}_k G(G^T \Phi_k^1 G + U)^{-1} G^T \tilde{\Phi}_k \tilde{F}_k^2 \tag{7.19}$$

where $\tilde{F}_k^1 = (I - G(G^T \Phi_k^1 G + U)^{-1} G^T \Phi_k^1) F$
$\tilde{F}_k^2 = (I - G(G^T \Phi_k^2 G + U)^{-1} G^T \Phi_k^2) F.$

2. *Suppose the RDE (7.17) is associated with a nonsingular spectral matrix $\Psi(z)$ and the ARE (7.16) has a strong solution Φ. Suppose also that both $\Phi_0^1 \geq \Phi$ and $\Phi_0^2 \geq \Phi$. Then if $\tilde{\Phi}_0 \geq 0$ it follows that $\tilde{\Phi}_k \geq 0$ for all $k \geq 0$.*

Proof: A more general version of the first difference equation which also accounts for perturbations in V was proven in the appendix of [23]. The second difference equation in item *1* can be obtained from the first simply by reversing the superscripts and then multiplication of the equation by -1.

Item *2* has been established in the nonnegative definite cost case for LQ control and Kalman filtering [9]. A generalization of this result to the broader class of spectral matrices we consider here follows from the discussion below.

Observe that with $\Phi_0^1 = \Phi$, $\Phi_k^1 = \Phi$ for all subsequent k. By hypothesis, $\Phi_0^2 \geq \Phi$ and hence $\tilde{\Phi}_0 \geq 0$. Next observe from (7.19) that since $G^T \Phi G + U > 0$ (which follows from the assumed spectral property), it follows that $\Phi_k^2 \geq \Phi$ for all subsequent k.

Suppose now that one is given any $\Phi_0^1 \geq \Phi$. It follows from reversing subscripts in the argument immediately above that $\Phi_k^1 \geq \Phi$ for all subsequent k. Since $G^T \Phi G + U > 0$, it follows that $G^T \Phi_k^1 G + U > 0$ which together with (7.19) implies that $\tilde{\Phi}_k \geq 0$ for all subsequent k. □

7.4 A Preliminary Convergence Result.

In the present section, some additional (and temporary) assumptions are adopted concerning the spectral matrix realization and the initial condition of the RDE. The main result of this section is stated in Lemma 7.4.1 and proven in the remainder of the section. This result concerns the convergence rate of the RDE when the associated spectral matrix realization has unit circle invariant zeros. This result is a weakened version of the main convergence result presented in 7.5, where it is shown how the additional assumptions introduced in Lemma 7.4.1 may be relaxed.

Properties *1* and *2* in the following lemma can be found in [4] and [15]. Observations of the $\mathcal{O}(\frac{1}{k})$ convergence rate have been made in a numerical study recorded in [102]. This convergence rate has been deduced for simple first order examples in [3], [15] and [22]. The worst-case RDE convergence rate given in item *3 a)* is stated in [4] in connection with the spectral factorization problem associated with the discrete positive real lemma (see [4]). However a detailed proof of this convergence rate is not given in [4].

In proving the following lemma, a novel and nontrivial completion of the proof of convergence is presented which initially follows much the same reasoning as that given in [4] and [15], but which then investigates the convergence mechanism in more detail so that the stated rates can be deduced.

Lemma 7.4.1 *Consider the following realization of a nonsingular discrete time spectral matrix*

$$\Psi(z) = U + G^T(z^{-1}I - F^T)^{-1}V(zI - F)^{-1}G, \tag{7.20}$$

which, in addition to assumptions **DA.1** *((F, G) is stabilizable) and* **DA.2** *(U is nonsingular), satisfies the following two assumptions:*

DA.3 *(F, G) is controllable.*

DA.4 *F is nonsingular.*

Let Φ be the strong solution of the associated algebraic Riccati equation (7.16); then provided $\Phi_0 > \Phi$, iterates of the associated Riccati difference equation (7.17) have the following properties:

1. *$\Phi_k > \Phi$.*
2. *$\lim_{k\to\infty} \Phi_k = \Phi$.*
3. *If the above realization of $\Psi(z)$ has an invariant zero on the unit circle, then there exist constants δ_1, δ_2 (depending on the realization of $\Psi(\cdot)$ and on Φ_0) with $\delta_1 \geq \delta_2 > 0$ such that:*

   a) For all $\epsilon > 0$, there exists a positive integer k_ϵ such that when $k \geq k_\epsilon$,

   $$\lambda_{\max}(\Phi_k - \Phi) \leq \frac{\delta_1 + \epsilon}{k}. \tag{7.21}$$

   b) For all $\eta \in (0, \delta_2)$, there exists a positive integer k_η such that when $k \geq k_\eta$,

   $$\lambda_{\max}(\Phi_k - \Phi) \geq \frac{\delta_2 - \eta}{k}. \tag{7.22}$$

In the next subsection the first step towards a proof of the above result is presented, which says that it is possible to describe RDE evolution in terms of a linear matrix difference equation (as described in [4] and [15]). Subsequently it is shown that upper and lower bounds for this difference equation can be found in terms of another, somewhat simpler linear matrix difference equation. A detailed study of the convergence behaviour of the simpler linear matrix difference equation gives rise to the convergence rate results which are stated in the above lemma.

7.4.1 A Linear Matrix Difference Equation describing RDE Evolution.

Under the additional assumptions adopted in Lemma 7.4.1, it is well known that the evolution of the RDE (7.17) can be described by that of a related linear matrix difference equation. This observation was made in both [4] and [15] but the behaviour of this difference equation was not studied in detail in these references.

Derivation of the Difference Equation for Δ_k^{-1}.

With the definitions $\Delta_k = \Phi_k - \Phi$ and

$$\tilde{F} = (I - G(G^T\Phi G + U)^{-1}G^T\Phi)F, \tag{7.23}$$

one can apply part *1* of Lemma 7.3.1 with $\Phi_k^2 = \Phi_k$, $\Phi_k^1 = \Phi$ and $\tilde{F}_k^1 = \tilde{F}$ to obtain

$$\Delta_{k+1} = \tilde{F}^T\Delta_k\tilde{F} - \tilde{F}^T\Delta_k G(G^T\Delta_k G + G^T\Phi G + U)^{-1}G^T\Delta_k\tilde{F}\ . \tag{7.24}$$

Suppose $\Delta_k > 0$, then since $N = U + G^T\Phi G > 0$ (due to the spectral property of $\Psi(z)$), the matrix inversion lemma may be applied to (7.24), revealing that

$$\Delta_{k+1}^{-1} = \tilde{F}^{-1}\Delta_k^{-1}\tilde{F}^{-T} + \tilde{F}^{-1}GN^{-1}G^T\tilde{F}^{-T}, \tag{7.25}$$

which has been derived in [4] and [15]. Invertibility of $\tilde{F}$ is a consequence of the invertibility of F and N; application of the matrix inversion lemma to (7.23) yields

$$\tilde{F}^{-1} = F^{-1}(I + GU^{-1}G^T\Phi). \tag{7.26}$$

Since $N^{-1} > 0$, (7.25) implies that $\Delta_{k+1} > 0$ whenever $\Delta_k > 0$. Thus our assumption that $\Delta_0 > 0$ ensures $\Delta_k > 0$ for all $k \geq 0$. This establishes item *1* in the statement of Lemma 7.4.1.

Observe that $\tilde{F}$ is of the form $\tilde{F} = F - GL$ (where $L = (G^T\Phi G + U)^{-1}G^T\Phi F$). It is a well known result that controllability of the pair (F, G) guarantees controllability of $(\tilde{F}, G)$, which in turn implies the controllability of $(\tilde{F}^{-1}, G)$. Since $(\tilde{F}^{-1}, G)$ is a controllable pair, so is the pair $(\tilde{F}^{-1}, GN^{-\frac{1}{2}})$. Hence the controllability Gramian for the latter pair satisfies

$$\tilde{W} = \sum_{j=0}^{n-1} \tilde{F}^{-j}GN^{-1}G^T(\tilde{F}^T)^{-j} > 0,$$

with n the dimension of the state space; i.e. $F \in \mathbb{R}^{n\times n}$. Direct iteration of the identity (7.25) then reveals that

$$\Delta_{k+n}^{-1} = \tilde{F}^{-n}\Delta_k^{-1}(\tilde{F}^T)^{-n} + \tilde{W}, \tag{7.27}$$

which is the linear matrix difference equation which we shall use in the remainder of this section as a means for studying the convergence behaviour of the RDE.

Remark: The above observations, leading to the linear matrix difference equation (7.27), have been made in [4] and [15]. As in Appendix III of [4] and in (4.14) and (4.15) of [15], convergence to zero of the iterates of (7.24) is based on the following observation:

$$\lambda_{\min}(\Delta_k^{-1}) \;\rightarrow\; \infty \;\Rightarrow\; \lambda_{\max}(\Delta_k) \rightarrow 0. \tag{7.28}$$

However, in contrast to the presentation of [4] and [15], the present exposition gives an explicit account of the divergent behaviour of $\lambda_{\min}(\Delta_k^{-1}) \;\rightarrow\; \infty$. A full description of the behaviour of (7.27) is not, to the author's knowledge, available in the literature. The purpose here is primarily to discuss its behaviour when $\tilde{F}$ has unit circle eigenvalues. In this case, $\tilde{F}^{-n}$ will also have unit circle eigenvalues, possibly with Jordan blocks of size greater than one. □

7.4.2 Bounding Sequences for the Linear Matrix Difference Equation.

In this subsection, we shall begin a study the linear matrix difference equation (7.27) by introducing a Jordan canonical form for the matrix $\tilde{F}^{-n}$. Subsequently, two sequences of matrices are defined which provide upper and lower bounds for iterates of the transformed linear matrix difference equation. These two sequences are defined in terms of another simpler linear matrix difference equation.

For convenience, we now introduce the the following definitions $\tilde{A} = \tilde{F}^{-n}$ and $\tilde{X}_j = \Delta_{jn}^{-1}$, in which case (7.27) reads

$$\begin{aligned} \tilde{X}_{j+1} &= \tilde{A}\tilde{X}_j\tilde{A}^T + \tilde{W}, && (7.29)\\ \tilde{X}_0 &= \Delta_0^{-1}. && (7.30) \end{aligned}$$

Review of the Real Jordan Form.

The following summarizes standard results concerning the real Jordan decomposition (see for example section 0.11 of [104]), which are introduced in the subsequent analysis of the above difference equation for $\tilde{X}_j$.

Any square matrix B with real coefficients can be expressed as

$$B = TAT^{-1}, \tag{7.31}$$

where T is a nonsingular similarity transformation and

$$A = \operatorname{diag}\{A_1, \ \cdots \ , A_p\} \tag{7.32}$$

and p is the number of real Jordan blocks. For each $q \in \{1, \dots, p\}$, A_q has one of the two forms described below.

In the first form, $A_q \in \mathbb{R}^{2n_q \times 2n_q}$ where $n_q \geq 1$ and

$$A_q = \begin{pmatrix} \Lambda_q & & & & & \\ I_2 & \Lambda_q & & & & \\ & I_2 & \Lambda_q & & & \\ & & \ddots & \ddots & & \\ & & & & \Lambda_q & \\ & & & & I_2 & \Lambda_q \end{pmatrix} \tag{7.33}$$

where

$$\Lambda_q = \begin{pmatrix} \sigma_q & \omega_q \\ -\omega_q & \sigma_q \end{pmatrix}. \tag{7.34}$$

In this case, $\sigma_q \pm j\omega_q$ is a pair of complex conjugate eigenvalues of B.

In the second form, $A_q \in \mathbb{R}^{n_q \times n_q}$ where $n_q \geq 1$ and

$$A_q = \begin{pmatrix} \lambda_q & & & & & \\ 1 & \lambda_q & & & & \\ & 1 & \lambda_q & & & \\ & & \ddots & \ddots & & \\ & & & & \lambda_q & \\ & & & & 1 & \lambda_q \end{pmatrix}. \tag{7.35}$$

where λ_q is a real eigenvalue of B.

Let $\tilde{A}$ have the real Jordan decomposition

$$\tilde{A} = TAT^{-1}, \tag{7.36}$$

where A has the structure described in (7.32). Since Φ is, by hypothesis, a strong solution of the algebraic Riccati equation, we know that $|\lambda_i(\tilde{F})| \leq 1$. It can be easily checked that $|\lambda_i(\tilde{A})| \geq 1$ is a consequence of this.

With T the transformation in (7.36), observe that one can express $\tilde{X}_j$ as $\tilde{X}_j = TX_jT^T$, where X_j are iterates defined by the equation

$$\begin{aligned} X_{j+1} &= AX_jA^T + W, && (7.37) \\ X_0 &= T^{-1}\tilde{X}_0T^{-T}. && (7.38) \end{aligned}$$

Here $W = T^{-1}\tilde{W}T^{-T} > 0$ and therefore $X_j > 0$ for all $j \geq 0$.

We next define a sequence of matrices $\{Y_j\}$ which *underbounds* $\{X_j\}$:

$$\begin{aligned} Y_{j+1} &= AY_jA^T + \lambda_{\min}(W)I, && (7.39) \\ Y_0 &= 0. && (7.40) \end{aligned}$$

It is trivial to show by induction that $0 < Y_j \leq X_j$ for all $j \geq 0$. Thus if we can show that $\{Y_j\}$ diverges, divergence of $\{X_j\}$ and $\{\tilde{X}_j\}$ follow.

Let $\mu > 0$ be chosen such that $\mu \geq \lambda_{\max}(X_1) = \lambda_{\max}(AX_0A^T + W)$, then the sequence of matrices Z_j defined below *overbounds* X_j:

$$Z_{j+1} = AZ_jA^T + \mu I \tag{7.41}$$
$$Z_0 = 0. \tag{7.42}$$

It is trivial to show by induction that $Z_j \geq X_j$ for all $j \geq 1$.

The question now remains as to how we can establish the convergence behaviour of these bounding sequences $\{Y_j\}$ and $\{Z_j\}$ and thus secure results on convergence of the RDE iterates. It should be noted that, except for scaling on the constant term, the sequences $\{Y_j\}$ and $\{Z_j\}$ evolve according to the *same* linear matrix difference equation. The behaviour of this difference equation is the subject of the next subsection.

7.4.3 Evolution of a Linear Matrix Difference Equation with Unit Circle Eigenvalues.

A linear matrix difference equation with Jordan structure.
Because of its significance in describing the convergence behaviour of the Riccati difference equation to its strong solution, we examine the behaviour of the following linear matrix difference equation:

$$X_q(m+1) = A_qX_q(m)A_q^T + I \tag{7.43}$$
$$X_q(0) = 0. \tag{7.44}$$

We assume that A_q has the first real Jordan form as described in (7.33) which corresponds to a complex conjugate pair of eigenvalues. Similar arguments to those which will be presented for this case can be used in the second case (7.35) (where λ is real) and are therefore not treated here.

It follows by direct iteration of (7.43) that for $m \geq 1$, $X_q(m) = S_q(m)$ where

$$S_q(m) = \sum_{l=0}^{m-1} A_q^l(A_q^T)^l. \tag{7.45}$$

The main objective of the present subsection is to study the behaviour of $S_q(m)$ as a function of m via a series of lemmas. This will lead to a proof of the preliminary convergence result which is presented in subsection 7.4.4. We focus here on Jordan blocks A_q which correspond to eigenvalues which are *on* the unit circle. It is the behaviour of iterates of this type that are the most important in establishing the convergence rate of RDEs associated with spectral matrices which have unit circle invariant zeros.

Remark:

For convenience, we review briefly some notation which is used to express the results which follow:

1) Let $f(l)$ be a scalar valued function and $U(l)$ be a matrix valued function, both of an integer variable l. We say that $U(l) = \mathcal{O}(f(l))$ if $\sigma_{\max}(U(l)) = \mathcal{O}(f(l))$, where the notation $\mathcal{O}(\cdot)$ applied to scalar functions has the standard definition.

2) Suppose M has an even number of rows and columns, consisting of a matrix of 2×2 matrix sub-blocks; for convenience we let $[M]_{ij} \in \mathbb{R}^{2\times 2}$ denote the $(i,j)^{\text{th}}$ 2×2 subblock of M. □

The following three lemmas together describe the *divergence* behaviour of $S_q(m)$. They do so only for Jordan blocks corresponding to a complex conjugate pair of unit circle eigenvalues (i.e. $\lambda \neq \pm 1$). Completely analogous results are available, however for the $\lambda = \pm 1$ case. The first lemma presents an approximation formula for the terms $A_q^l (A_q^T)^l$ which appear in the definition of $S_q(m)$. The second lemma presents an approximation formula for $S_q(m)$ itself, based on the approximation formula for $A_q^l(A_q^T)^l$. Proofs of the first two of the lemmas presented below can be found in Appendix J. The third lemma is the key as far as the development in this section is concerned. It gives upper and lower bounds for the *divergence* rate of $S_q(m)$. A proof of this lemma is given in the present section which is based on the approximation formula for $S_q(m)$ given in the second lemma.

Lemma 7.4.2 *Let A_q be a Jordan block of size $2n_q \times 2n_q$ which has the form given in (7.33), corresponding to a complex conjugate pair of unit circle eigenvalues. Then for sufficiently large l, one has the following identity:*

$$A_q^l (A_q^T)^l = C(l) + P(l) = C(l)(I + \mathcal{O}(l^{-1})) \tag{7.46}$$

where

$$C(l) = \begin{pmatrix} I_2 & l\Lambda_q & \frac{l^2}{2}\Lambda_q^2 & \cdots & \frac{l^{n_q-1}}{(n_q-1)!}\Lambda_q^{n_q-1} \\ l(\Lambda_q^T) & l^2 I_2 & \frac{l^3}{2}\Lambda_q & & \vdots \\ \frac{l^2}{2}(\Lambda_q^T)^2 & \frac{l^3}{2}(\Lambda_q^T) & \frac{l^4}{4} I_2 & & \\ \vdots & & & \ddots & \\ \frac{l^{n_q-1}}{(n_q-1)!}(\Lambda_q^T)^{n_q-1} & & & & \frac{l^{2n_q-2}}{(n_q-1)!(n_q-1)!} I_2 \end{pmatrix} \tag{7.47}$$

or equivalently

$$[C(l)]_{ij} = \begin{cases} \frac{l^{i+j-2}}{(i-1)!(j-1)!}\Lambda_q^{j-i} & \text{if } i \leq j \\ [C(l)]_{ji}^T & \text{if } i > j \end{cases}, \tag{7.48}$$

and

$$[P(l)]_{ij} = \begin{cases} \mathcal{O}(l^{i+j-3}) & \text{if } i+j \geq 3 \\ 0_2 & \text{if } i = j = 1 \end{cases}. \tag{7.49}$$

Proof: Refer to appendix J.1. □

Lemma 7.4.3 *Let A_q be a Jordan block of size $2n_q \times 2n_q$ which has the form given in (7.33), corresponding to a complex conjugate pair of unit circle eigenvalues. Then with $S_q(m)$ defined in (7.45), the following identity holds:*

$$S_q(m) = D(m) + G(m) = D(m)(I + \mathcal{O}(m^{-1})) \tag{7.50}$$

where

$$D(m) = \begin{pmatrix} mI_2 & \frac{m^2}{2}\Lambda_q & \frac{m^3}{6}\Lambda_q^2 & \cdots & \frac{m^{n_q}}{(n_q-1)!n_q}\Lambda^{(n_q-1)} \\ \frac{m^2}{2}(\Lambda_q^T) & \frac{m^3}{3}I_2 & \frac{m^4}{8}\Lambda_q & & \vdots \\ \frac{m^3}{6}(\Lambda_q^T)^2 & \frac{m^4}{8}(\Lambda_q^T) & \frac{m^5}{20}I_2 & & \\ \vdots & & & \ddots & \\ \frac{m^{n_q}}{(n_q-1)!n_q}(\Lambda_q^T)^{n_q-1} & \cdots & & & \frac{m^{2n_q-1}}{(n_q-1)!(n_q-1)!(2n_q-1)}I_2 \end{pmatrix},$$

or equivalently

$$[D(m)]_{ij} = \begin{cases} \frac{m^{i+j-1}}{(i-1)!(j-1)!(i+j-1)}(\Lambda_q^T)^{i-j} & \text{if } i \geq j \\ [D(m)]_{ji}^T & \text{if } i < j \end{cases} \tag{7.51}$$

and

$$[G(m)]_{ij} = \begin{cases} \mathcal{O}(m^{i+j-2}) & \text{if } i+j \geq 3 \\ 0_2 & \text{if } i = j = 1 \end{cases}. \tag{7.52}$$

Moreover, there exists a matrix $\Theta \in \mathbb{R}^{2n_q \times 2n_q}$ such that $\Theta > 0$ and

$$D(m) = mH^T(m)\Theta H(m) \tag{7.53}$$

where

$$H(m) = \operatorname{diag}\{I_2, \ m\Lambda_q, \ \cdots \ , m^{(n_q-1)}\Lambda_q^{(n_q-1)}\}. \tag{7.54}$$

Proof: Refer to appendix J.2. □

Lemma 7.4.4 *Let $A_q \in \mathbb{R}^{n_q \times n_q}$ be a Jordan block of the form given in (7.33) which corresponds to a complex conjugate pair of eigenvalues on the unit circle. Let $S_q(m)$ be defined as in (7.45). Then there exists a constant $\zeta > 0$ such that*

1. *For all $\eta \in (0, \zeta)$ such that $\zeta > \eta > 0$, there exists an integer $m_\eta \geq 0$ such that when $m > m_\eta$,*
$$\lambda_{\min}(S_q(m)) \leq m\frac{2n_q}{\zeta - \eta}. \tag{7.55}$$
2. *For all $\epsilon > 0$, there exists an integer $m_\epsilon \geq 0$ such that when $m > m_\epsilon$,*
$$\lambda_{\min}(S_q(m)) \geq \frac{m}{\zeta + \epsilon}. \tag{7.56}$$

Proof: Recall first that $S_q(m) = mH^T(m)\Theta H(m) + G(m)$, where $H(m)$, $G(m)$ and Θ

are as described in Lemma 7.4.3. Since $H(m)$ is invertible, it follows that

$$S_q(m) = mH^T(m)\{\Theta + W(m)\}H(m) \tag{7.57}$$

where

$$W(m) = H^{-T}(m)\frac{G(m)}{m}H^{-1}(m). \tag{7.58}$$

It can also be verified fairly simply that $W(m) = \mathcal{O}(m^{-1})$.

That $S_q(m)$ is always a positive definite matrix can be seen from its definition in (7.45). In order to describe its eigenvalues as a function of m, we investigate those of $S_q^{-1}(m)$. Now recall that for any positive definite matrix M, if $\lambda_{\max}(M)$ is the maximum eigenvalue of M then the minimum eigenvalue of M^{-1} is $\lambda_{\min}(M^{-1}) = \lambda_{\max}^{-1}(M)$.

Since $S_q(m)$ is positive definite, it follows from (7.57) that $\Theta + W(m)$ is positive definite. Observe next that since $W(m) = \mathcal{O}(m^{-1})$ and $\Theta > 0$,

$$(\Theta + W(m))^{-1} \;=\; \Theta^{-\frac{1}{2}}(I + \mathcal{O}(m^{-1}))^{-1}\Theta^{-\frac{1}{2}}. \tag{7.59}$$

Note next the following identity:

$$(I + \mathcal{O}(m^{-1}))^{-1} \;=\; I + \mathcal{O}(m^{-1}), \tag{7.60}$$

which follows from two facts: firstly $(I + \mathcal{O}(m^{-1}))^{-1} - I = (I + \mathcal{O}(m^{-1}))^{-1}\mathcal{O}(m^{-1})$, and secondly $\sigma_{\max}\left((I + \mathcal{O}(m^{-1}))^{-1}\right) = \sigma_{\min}\left(I + \mathcal{O}(m^{-1})\right) \leq \sigma_{\max}\left(I + \mathcal{O}(m^{-1})\right) = 1 + \mathcal{O}(m^{-1})$.

From (7.60), it follows that

$$(\Theta + W(m))^{-1} \;=\; \Theta^{-1} + \mathcal{O}(m^{-1}). \tag{7.61}$$

Inverting (7.57) and employing (7.61) reveals that

$$mS_q^{-1}(m) = H^{-1}(m)\left\{\Theta^{-1} + \mathcal{O}(m^{-1})\right\}H^{-T}(m) \tag{7.62}$$

and since $H^{-1}(m) = \mathcal{O}(1)$,

$$mS_q^{-1}(m) = H^{-1}(m)\Theta^{-1}H^{-T}(m) + \mathcal{O}(m^{-1}). \tag{7.63}$$

With M a nonnegative definite matrix of dimension n_M, recall the standard identity

$$\lambda_{\max}(M) \leq \operatorname{trace}\{M\} \leq n_M\lambda_{\max}(M). \tag{7.64}$$

Applying this result to (7.63) reveals that

$$\lambda_{\max}(mS_q^{-1}(m)) \leq \operatorname{trace}\left\{H^{-1}(m)\Theta^{-1}H^{-T}(m)\right\} + \mathcal{O}(m^{-1}) \leq 2n_q\lambda_{\max}(mS_q^{-1}(m)). \tag{7.65}$$

Observe that

$$\left[H^{-1}(m)\Theta^{-1}H^{-T}(m)\right]_{ij} = \frac{1}{m^{2-i-j}}\Lambda_q^{1-i}\left[\Theta^{-1}\right]_{ij}(\Lambda_q^T)^{1-j} \tag{7.66}$$

and as a result that

$$\operatorname{trace}\left\{H^{-1}(m)\Theta^{-1}H^{-T}(m)\right\} = \operatorname{trace}\left\{\left[\Theta^{-1}\right]_{11}\right\} + \mathcal{O}(m^{-2}). \tag{7.67}$$

With the definition $\zeta = \operatorname{trace}\{[\Theta^{-1}]_{11}\}$, note firstly from (7.65) and (7.67) that

$$\lambda_{\max}(mS_q^{-1}(m)) \leq \zeta + \mathcal{O}(m^{-1}) \tag{7.68}$$

and secondly that

$$\lambda_{\max}(mS_q^{-1}(m)) \geq \frac{\zeta}{2n_q} + \mathcal{O}(m^{-1}). \tag{7.69}$$

The stated results follow immediately from (7.68) and (7.69). □

7.4.4 Proof of the Preliminary Convergence Result.

In the light of the conclusions from the last section, we now return to describing the behaviour of the sequences $\{Y_j\}$ and $\{Z_j\}$ which were introduced in section 7.4.2. Recall from the discussion in subsection 7.4.2 that A is in Jordan form, $A = \operatorname{diag}\{A_1, \cdots, A_p\}$, and that $|\lambda_i(A)| \geq 1$. It follows therefore that $|\lambda_i(A_q)| \geq 1$ for $q \in \{1, \ldots, p\}$.

Direct calculation of Y_j yields the formula:

$$\begin{aligned} Y_j &= \lambda_{\min}(W)\left\{\sum_{l=0}^{j-1} A^l (A^T)^l\right\} &(7.70)\\ &= \lambda_{\min}(W)\operatorname{diag}\{S_1(j), \cdots, S_p(j)\}, &(7.71)\end{aligned}$$

where $S_q(j) = \sum_{l=0}^{j-1} A_q^l (A_q^T)^l$ for $q \in \{1, \ldots, p\}$. Thus the set of eigenvalues of Y_j is simply the union of all the eigenvalues of $S_q(j)$ for all q, scaled by $\lambda_{\min}(W)$. In particular note that

$$\lambda_{\min}(Y_j) = \lambda_{\min}(W) \min_{q \in \{1,\ldots,p\}} \lambda_{\min}(S_q(j)). \tag{7.72}$$

It follows from the properties of $S_q(j)$ described in the previous subsection that there must exist an integer $j_0 > 1$ such that when $j \geq j_0$,

$$\lambda_{\min}(Y_j) = \lambda_{\min}(W)\lambda_{\min}(S_r(j)) \tag{7.73}$$

where r corresponds to a Jordan block A_r of A with unity eigenvalue. From Lemma 7.4.4, it follows that there exists a constant $\zeta_r > 0$ such that for all $\hat{\epsilon} > 0$, there exists an integer j_1 such that when $j \geq j_1$,

$$\lambda_{\min}(Y_j) \geq \lambda_{\min}(W)\frac{j}{\zeta_r + \hat{\epsilon}}. \tag{7.74}$$

Next recall from the discussion in subsection 7.4.1 that $\tilde{X}_j = \Delta_{jn}^{-1} = TX_jT^T$. Hence provided $j \geq \max\{j_0, j_1\}$,

$$\lambda_{\min}(\tilde{X}_j) \geq \sigma^2_{\min}(T)\lambda_{\min}(X_j) \tag{7.75}$$

$$\geq \sigma_{\min}^2(T)\lambda_{\min}(Y_j) \tag{7.76}$$
$$\geq \sigma_{\min}^2(T)\lambda_{\min}(W)\frac{j}{\zeta_r+\hat{\epsilon}}. \tag{7.77}$$

Note therefore that there exists a constant $\alpha > 0$ such that for any $\epsilon > 0$, there exists an integer j_2 such that when $j \geq j_2$,

$$\lambda_{\max}(\Delta_{jn}) \leq \frac{\alpha+\epsilon}{jn}. \tag{7.78}$$

It then follows immediately that $\lambda_{\max}(\Delta_{jn}) \to 0$, this ensuring that $\Phi_{jn} \to \Phi$.

Since for $s \in \{1,\dots,n-1\}$ it is known that $\Delta_s > 0$, identical reasoning to the above can be applied to deduce that $\lambda_{\max}(\Delta_{jn+s}) \to 0$ and therefore that $\Phi_{jn+s} \to \Phi$. It follows that $\lambda_{\max}(\Delta_k) \to 0$. Note also that for each $s \in \{0,1,\dots,n-1\}$, an inequality of the form (7.78) is available for Φ_{jn+s}. Combining these inequalities yields part *3 a)* of the preliminary convergence result in Lemma 7.4.1.

In an identical manner to that employed in investigating Y_j, one can deduce the following expression for Z_j:

$$Z_j = \mu\text{diag}\{S_1(j), \quad\cdots\quad, S_p(j)\}. \tag{7.79}$$

Observe that

$$\lambda_{\min}(Z_j) = \mu \min_{q\in\{1,\dots,p\}} \lambda_{\min}(S_q(j)). \tag{7.80}$$

Reasoning identical to that used in the Y_j case, yields the existence of an integer $j_3 > 1$ such that when $j \geq j_3$,

$$\lambda_{\min}(Z_j) = \mu\lambda_{\min}(S_r(j)). \tag{7.81}$$

where r corresponds to a Jordan block A_r of A with unity eigenvalue. From Lemma 7.4.4, it follows that there exists a constant $\zeta_r > 0$ such that for all $\eta \in (0,\zeta_r)$, there exists an integer j_4 such that $j \geq j_4$ implies that

$$\lambda_{\min}(Z_j) \leq \mu j\frac{2n_r}{\zeta_r-\hat{\eta}}, \tag{7.82}$$

where n_r is the size of the Jordan block A_r.

Since $Z_j \geq X_j$ for all $j \geq 1$, note that $\lambda_{\min}(Z_j) \geq \lambda_{\min}(X_j)$. Hence provided $j \geq \max\{j_3, j_4\}$,

$$\lambda_{\min}(\tilde{X}_j) \leq \sigma_{\max}^2(T)\lambda_{\min}(X_j) \tag{7.83}$$
$$\leq \sigma_{\max}^2(T)\lambda_{\min}(Z_j) \tag{7.84}$$
$$\leq \sigma_{\max}^2(T)\mu j\frac{2n_r}{\zeta_r-\hat{\eta}}. \tag{7.85}$$

Note therefore that there exists a constant $\beta > 0$ such that for any $\beta > \eta > 0$, there exists an integer j_5 such that when $j \geq j_5$,

$$\lambda_{\max}(\Delta_{jn}) \geq \frac{\beta - \eta}{jn}. \tag{7.86}$$

Recall that for $s \in \{0, 1, \ldots, n-1\}$, it is true that $\Delta_s > 0$. An inequality analogous to (7.86) but for Δ_{jn+s} is available for each s. Combining these inequalities yields part *3 b)* of the preliminary convergence result of Lemma 7.4.1. □

7.5 The Main Convergence Result.

In this section, we return to considering spectral matrices realized as follows

$$\Psi(z) = U + G^T(z^{-1}I - F^T)^{-1}V(zI - F)^{-1}G, \tag{7.87}$$

which satisfy assumptions **DA.1** ((F, G) stabilizable) and **DA.2** (U nonsingular) but which may not satisfy the stronger assumptions **DA.3** and **DA.4** which were required in the preliminary convergence result presented in the last section. Recall that our main objective is to study the convergence behaviour of the following RDE

$$\Phi_{k+1} = F^T\left(\Phi_k - \Phi_k G(U + G^T\Phi_k G)^{-1}G^T\Phi_k\right)F + V \tag{7.88}$$

towards the strong solution of the discrete algebraic Riccati equation

$$\Phi = F^T\left(\Phi - \Phi G(U + G^T\Phi G)^{-1}G^T\Phi\right)F + V. \tag{7.89}$$

The realization of the spectral matrix associated with the above RDE may have unit circle invariant zeros which arise due to nonminimal modes, transmission zeros, or any combination of these.

Recall that convergence has previously been established for linear-quadratic control and Kalman filtering problems (see [15] and [22]) and for the spectral factorization problem associated with the discrete positive real lemma (see [4]). In [4], assumption **DA.3** is made (that (F, G) is controllable), whilst assumption **DA.4** (that F is nonsingular) is relaxed. Convergence rates are stated in [4] but a proof is not given. In [15], both assumptions **DA.3** and **DA.4** are made and, except for a scalar example, no convergence rates are given. Following on from [15], it is shown in [22] (albeit by different means to those proposed here) that the condition (F, G) controllable can be relaxed to (F, G) stabilizable and that singular F matrices can be accommodated. However, except for a scalar example, no convergence rates are stated in [22] either.

New results are presented below concerning the *rate* at which the iterates of the Riccati difference equation converge to the strong solution of the DARE (7.89) under assumptions **DA.1** and **DA.2** only. The convergence result which was derived in the previous section under the additional assumptions **DA.3**, $\Phi_0 > \Phi$ and **DA.4** is central to the proof of the following result. The proof proceeds via a successive relaxation of these additional conditions to allow (F, G) stabilizable, $\Phi_0 \geq \Phi$, and finally also F singular.

Theorem 7.5.1 *Given a realization (7.87) of a discrete time spectral matrix $\Psi(z)$ satisfying assumptions* **DA.1** *and* **DA.2** *, along with the strong solution Φ of the associated algebraic Riccati equation (7.89), then provided $\Phi_0 \geq \Phi$, iterates of (7.88) have the following properties:*

1. $\Phi_k \geq \Phi$.
2. $\lim_{k\to\infty} \Phi_k = \Phi$.
3. *If $\Psi(z)$ has an invariant zero on the unit circle, then there exist constants κ_1, κ_2 (depending on the realization of $\Psi(\cdot)$ and on Φ_0) with $\kappa_1 \geq \kappa_2 > 0$ such that:*

   a) For all $\epsilon > 0$, there exists a k_ϵ such that when $k \geq k_\epsilon$

$$\lambda_{\max}(\Phi_k - \Phi) \leq \frac{\kappa_1 + \epsilon}{k}. \tag{7.90}$$

   b) Suppose that $\Phi_0 > \Phi$, then for all $\eta \in (0, \kappa_2)$, there exists a k_η such that when $k \geq k_\eta$

$$\lambda_{\max}(\Phi_k - \Phi) \geq \frac{\kappa_2 - \eta}{k}. \tag{7.91}$$

Remarks:
a) Part *3 a)* of the above theorem reports a worst-case $\frac{1}{k}$ convergence rate in the case of unit-circle invariant zeros. Part *3 b)* says that, with the exclusion of (non-generic) cases where $\Phi_0 - \Phi$ is singular, the convergence rate can be no better than $\frac{1}{k}$.
b) If the spectral matrix has no invariant zeros on the unit circle, the convergence rate will in general be *exponential.* For example, it is shown in Theorem 5.4 of [13] that for realizations of spectral matrices with nonnegative definite cost having (F, G) stabilizable and (F, V) detectable, Φ_k converges exponentially to Φ. In other words, there exist constants A, κ_3 such that $1 > \kappa_3 > 0$ and

$$\lambda_{\max}(\Phi_k - \Phi) \leq A\kappa_3^k. \tag{7.92}$$

Similar statements are made in section IV of [4] for the Riccati difference equation associated with the discrete positive real lemma. □

7.5.1 Relaxing assumption DA.3 (that (F, G) is controllable.)

The aim of this subsection is to establish the following result:

First strengthening of Lemma 7.4.1:
With the additional assumptions **DA.4** (F is nonsingular)and $\Phi_0 > \Phi$, the statements

in Theorem 7.5.1 hold.

In [22], assumption **DA.3** was relaxed to prove convergence of the RDE using a sequence of perturbations on the original problem. The emphasis in the present paper is to investigate the *structure* of RDE iterates associated with the stable uncontrollable modes of (F, G). These observations give rise to statements concerning the convergence rate which were not made in [22].

We assume now that (F, G) is stabilizable and that, without loss of generality,

$$F = \begin{pmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{pmatrix} \tag{7.93}$$

$$G = \begin{pmatrix} G_1 \\ 0 \end{pmatrix} \tag{7.94}$$

where $|\lambda_i(F_{22})| < 1$ and (F_{11}, G_1) is a controllable pair. We partition Φ_k and V conformally:

$$\Phi_k = \begin{pmatrix} \Phi_{11}^k & \Phi_{12}^k \\ {\Phi_{12}^k}^T & \Phi_{22}^k \end{pmatrix} \tag{7.95}$$

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{12}^T & V_{22} \end{pmatrix}. \tag{7.96}$$

Expression of the RDE (7.88) in terms of this partitioning reveals that Φ_{11}^k satisfies the Riccati difference equation

$$\Phi_{11}^{k+1} = F_{11}^T \left(\Phi_{11}^k - \Phi_{11}^k G_1 (U + G_1^T \Phi_{11}^k G_1)^{-1} G_1^T \Phi_{11}^k \right) F_{11} + V_{11}. \tag{7.97}$$

With conformal partitioning of Φ (the strong solution of (7.89)), it can be readily shown that Φ_{11} is a solution of the algebraic Riccati equation

$$\Phi_{11} = F_{11}^T \left(\Phi_{11} - \Phi_{11} G_1 (U + G_1^T \Phi_{11} G_1)^{-1} G_1^T \Phi_{11} \right) F_{11} + V_{11}. \tag{7.98}$$

Moreoever, Φ_{11} is also a strong solution of this equation, as is now shown. Observe that one can write

$$\begin{aligned} \tilde{F} &= (I - G(G^T \Phi G + U)^{-1} G^T \Phi) F \\ &= \begin{pmatrix} \tilde{F}_{11} & (I - G_1 N_{11}^{-1} G_1^T \Phi_{11}) F_{12} - G_1 N_{11}^{-1} G_1^T \Phi_{12} F_{22} \\ 0 & F_{22} \end{pmatrix}, \end{aligned} \tag{7.99}$$

where

$$\tilde{F}_{11} = \left(I - G_1 N_{11}^{-1} G_1^T \Phi_{11} \right) F_{11}, \tag{7.100}$$

and $N_{11} = N = U + G_1^T \Phi_{11} G_1$. Since Φ is a strong solution of the DARE (7.89), it follows from (7.99) than $\tilde{F}_{11}$ must have all its eigenvalues in the closed unit circle.

Recall that for the moment, we maintain the assumption that F is invertible, from which it follows that F_{11} is also invertible. Since we also assume that $\Phi_0 > \Phi$ and therefore that $\Delta_0 > 0$, it follows that $\Delta_{11}^0 > 0$. Since (F_{11}, G_1) is controllable, we can apply item *3 a)* of Lemma 7.4.1 to deduce that iterates of the reduced-order RDE (7.97) satisfy

$$\Delta_{11}^k = \mathcal{O}(\frac{1}{k}). \tag{7.101}$$

It can also be shown using the RDE (7.88) that the partitions Φ_{12}^k of the iterates Φ_k satisfy

$$\Phi_{12}^{k+1} = (\tilde{F}_{11}^k)^T \Phi_{12}^k F_{22} + W_{12}^k \tag{7.102}$$

where

$$\begin{aligned} \tilde{F}_{11}^k &= \left(I - G_1 (N_{11}^k)^{-1} G_1^T \Phi_{11}^k\right) F_{11}, & (7.103)\\ N_{11}^k &= U + G_1^T \Phi_{11}^k G_1, & (7.104)\\ W_{12}^k &= (\tilde{F}_{11}^k)^T \Phi_{11}^k F_{12} + V_{12}. & (7.105) \end{aligned}$$

Recall from (7.101) that $\Phi_{11}^k = \Phi_{11} + \mathcal{O}(\frac{1}{k})$. It follows from (7.104) that $N_{11}^k = N_{11} + \mathcal{O}(\frac{1}{k})$, from (7.103) that $\tilde{F}_{11}^k = \tilde{F}_{11} + \mathcal{O}(\frac{1}{k})$ and hence that $W_{12}^k = W_{12} + \mathcal{O}(\frac{1}{k})$ where $W_{12} = \tilde{F}_{11}^T \Phi_{11} F_{12} + V_{12}$. By hypothesis, there exists a solution Φ_{12} of the algebraic equation

$$\Phi_{12} = \tilde{F}_{11}^T \Phi_{12} F_{22} + W_{12}. \tag{7.106}$$

Subtracting this equation from (7.102) and simultaneously adding and subtracting the term $(\tilde{F}_{11}^k)^T \Phi_{12} F_{22}$ yields the equation

$$\Delta_{12}^{k+1} = (\tilde{F}_{11}^k)^T \Delta_{12}^k F_{22} + (\tilde{F}_{11}^k - \tilde{F}_{11})^T \Phi_{12} F_{22} + W_{12}^k - W_{12}. \tag{7.107}$$

We shall now summarize some convergence results of linear difference equations of the above form.

Convergence Rate for a Time Varying Linear Difference Equation.

As will become apparent, the following result gives convergence rates for difference equations with time varying coefficients such as (7.107). The proof of this result, presented in Appendix K, combines a number of results from the theory of linear difference equations. Neither a statement of this result nor a result from which it could be simply deduced could be found in the literature.

Lemma 7.5.1 *Let $\{\Upsilon_k\}$ be a bounded sequence of matrices defined for $k \geq 0$. Consider the linear matrix difference equation with (possibly nonsquare) iterates Ξ_k having a finite initial condition Ξ_0:*

$$\Xi_{k+1} = A_k \Xi_k B_k + \Upsilon_k. \tag{7.108}$$

Suppose the (square) matrix sequences $\{A_k\}$ and $\{B_k\}$ are bounded for all $k \geq 0$ and satisfy $A_k \to A$ and $B_k \to B$, where $|\lambda_i(A)\lambda_j(B)| < 1$ for all i and j.
Then if $\Upsilon_k = \mathcal{O}(\frac{1}{k})$, it follows that

$$\Xi_k = \mathcal{O}(\frac{1}{k}). \tag{7.109}$$

Proof: Refer to appendix K □

With the above result in mind, we now return to the linear difference equation for Δ_{12}^k given in (7.107). Recall that $\tilde{F}_{11}$ has all eigenvalues in the closed unit circle. Observe that F_{22} is stable. It follows that $|\lambda_i(\tilde{F}_{11})\lambda_j(F_{22})| < 1$. In the statement of Lemma 7.5.1, we now identify Ξ_k with Δ_{12}^k, A_k with $(\tilde{F}_{11}^k)^T$, B_k with F_{22} and Υ_k with the remaining terms in (7.107), which can be easily shown to be $\mathcal{O}(\frac{1}{k})$. We now apply Lemma 7.5.1 to (7.107) to conclude that

$$\Delta_{12}^k = \mathcal{O}(\frac{1}{k}). \tag{7.110}$$

Note that examination of the $(2,2)$ partition of the RDE (7.88) reveals the following iteration:

$$\Phi_{22}^{k+1} = F_{22}^T \Phi_{22}^k F_{22} + S_{22}^k \tag{7.111}$$

where

$$\begin{aligned} S_{22}^k &= F_{12}^T \Phi_{11}^k \tilde{F}_{12}^k + (\tilde{F}_{12}^k)^T \Phi_{12}^k F_{22} \\ &\quad + F_{22}^T (\Phi_{12}^k)^T \left(\tilde{F}_{12}^k - G_1 (N_{11}^k)^{-1} G_1^T \Phi_{12}^k F_{22}\right) + V_{22}, \end{aligned} \tag{7.112}$$

$$\tilde{F}_{12}^k = \left(I - G_1 (N_{11}^k)^{-1} G_1^T \Phi_{11}^k\right) F_{12}. \tag{7.113}$$

Recall that by hypothesis, there exists a solution Φ_{22} of the equation

$$\Phi_{22} = F_{22}^T \Phi_{22} F_{22} + S_{22} \tag{7.114}$$

where S_{22} is given by taking the limit of (7.112) as $k \to \infty$. Subtracting the above equation from (7.111) yields the equation

$$\Delta_{22}^{k+1} = F_{22}^T \Delta_{22}^k F_{22} + S_{22}^k - S_{22}. \tag{7.115}$$

From (7.101) and (7.110) it follows that $S_{22}^k = S_{22} + \mathcal{O}(\frac{1}{k})$. Since F_{22} is stable, we can apply Lemma 7.5.1 with $A_k = F_{22}^T$, $B_k = F_{22}$ and $\Upsilon_k = S_{22}^k - S_{22}$ to conclude that

$$\Delta_{22}^k = \mathcal{O}(\frac{1}{k}). \tag{7.116}$$

Since $\Delta_{ij}^k = \mathcal{O}(\frac{1}{k})$ for each partition of Δ_k, it follows that $\Delta_k = \mathcal{O}(\frac{1}{k})$. This establishes the worst-case convergence result in item *3 a)*.

We now establish the best-case result in item *3 b)*. Choose any Φ_0 such that $\Delta_0 > 0$ and note therefore that $\Delta_{11}^0 > 0$. For convenience, we now recount the difference equation for Φ_{11}^k which was introduced earlier:

$$\Phi_{11}^{k+1} = F_{11}^T \left(\Phi_{11}^k - \Phi_{11}^k G_1 (U + G_1^T \Phi_{11}^k G_1)^{-1} G_1^T \Phi_{11}^k\right) F_{11} + V_{11}. \tag{7.117}$$

Recall that Φ is a strong solution of the DARE (7.89) in the sense that the eigenvalues of the following matrix are either inside or on the unit circle:

$$\tilde{F} = \begin{pmatrix} \tilde{F}_{11} & (I - G_1 N_{11}^{-1} G_1^T \Phi_{11}) F_{12} - G_1 N_{11}^{-1} G_1^T \Phi_{12} F_{22} \\ 0 & F_{22} \end{pmatrix}. \tag{7.118}$$

Note that all unit circle eigenvalues of $\tilde{F}$ must be eigenvalues of $\tilde{F}_{11}$ since F_{22} is stable by hypothesis. Note that we can apply item *3 b)* of Lemma 7.4.1 to the RDE (7.88) to deduce that there exists a $\delta_2 > 0$ such that for all $\eta \in (0, \delta_2)$, there exists a k_η such that when $k \geq k_\eta$,

$$\lambda_{\max}(\Phi_{11}^k - \Phi_{11}) \geq \frac{\delta_2 - \eta}{k}. \tag{7.119}$$

Note now that the positive-definite matrix Δ_{11}^k is a partition of the larger positive-definite matrix Δ_k and hence that

$$\lambda_{\max}(\Phi_k - \Phi) \geq \lambda_{\max}(\Phi_{11}^k - \Phi_{11}) \geq \frac{\delta_2 - \eta}{k}, \tag{7.120}$$

provided $k \geq k_\eta$. This establishes item *3 b)*.

7.5.2 Relaxing the assumption: $\Phi_0 > \Phi$.

The aim of this subsection is to establish the following result:

Second strengthening of Lemma 7.4.1:
With the additional assumption **DA.4**, the statements in Theorem 7.5.1 hold.

Suppose now that $\Phi_0 \geq \Phi$ but not $\Phi_0 > \Phi$. It is well known that the difference equation (7.24) for Δ_k holds also when Δ_k is singular. In particular, from item *2* of Lemma 7.3.1, it follows that $\Delta_k \geq 0$ for all $k \geq 0$ which establishes item *1*.

Suppose we have any $\bar{\Phi}_0$ such that $\bar{\Phi}_0 \geq \Phi_0 \geq \Phi$ and $\bar{\Phi}_0 > \Phi$. From item *2* of Lemma 7.3.1 it follows that $\bar{\Phi}_k \geq \Phi_k \geq \Phi$ for all $k \geq 0$ (where $\bar{\Phi}_k$ are iterates of the RDE with initial condition $\bar{\Phi}_0$). Since $\bar{\Phi}_k \geq \Phi_k$ and the convergence of $\{\bar{\Phi}_k\}$ is guaranteed by item *2* of the first strengthening of Lemma 7.4.1, item *2* in the theorem statement is established.

Item *3 a)* in the first strengthening of Lemma 7.4.1 establishes a worst-case bound for the convergence rate of $\{\bar{\Phi}_k\}$ which by virtue of the above observations guarantees the same convergence rate for $\{\Phi_k\}$. Since the restriction $\Phi_0 > \Phi$ is maintained in item *3 b)* of the theorem, clearly the best-case convergence result of item *3 b)* in the first strengthening of Lemma 7.4.1 remains.

7.5.3 Relaxing assumption DA.4 (that F is nonsingular).

The aim of this subsection is to establish the following result:

Final strengthening of Lemma 7.4.1:

Without any further assumptions, the statements in Theorem 7.5.1 hold. □

If F is singular then $\tilde{F} = (I - GN^{-1}G^T\Phi)F$ will be also (recall that $N = U + G^T\Phi G > 0$). Suppose it has a Jordan canonical form $\tilde{F} = M^{-1}\bar{F}M$ where

$$\bar{F} = \mathrm{diag}\left\{\bar{F}_{nz} \quad \bar{F}_z\right\} \tag{7.121}$$

and $\bar{F}_{nz}$ and $\bar{F}_z$ are block-diagonal and contain Jordan blocks corresponding to the non-zero and zero eigenvalues of $\tilde{F}$, respectively. Let G in this basis be partitioned as follows: $\bar{G}^T = (MG)^T = \begin{pmatrix}\bar{G}_{nz}^T & \bar{G}_z^T\end{pmatrix}$.

The difference equation for Δ_k given in (7.24) can now be expressed in the coordinate basis introduced above with the definition $\bar{\Delta}_k = M^{-T}\Delta_k M^{-1}$:

$$\bar{\Delta}_{k+1} = \bar{F}^T\bar{\Delta}_k\bar{F} - \bar{F}^T\bar{\Delta}_k\bar{G}(\bar{G}^T\bar{\Delta}_k\bar{G} + G^T\Phi G + U)^{-1}\bar{G}^T\bar{\Delta}_k\bar{F}. \tag{7.122}$$

It should be noted that this equation still holds under the assumptions adopted here (i.e. those stated in Theorem 7.5.1).

Let $\bar{F}_q$ be any Jordan block of $\bar{F}$ corresponding to a zero eigenvalue:

$$\bar{F}_q = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & \ddots & 0 & \\ & & & 1 & 0 \end{pmatrix}. \tag{7.123}$$

We now investigate the RDE evolution in the subspace corresponding to this Jordan block. It can be shown fairly easily via (7.122) that

$$(\bar{\Delta}_k)_q = \mathrm{diag}\{D_k \quad 0_k\} \tag{7.124}$$

where $(\bar{\Delta}_k)_q$ is the $n_q \times n_q$ diagonal subblock of $\bar{\Delta}_k$ corresponding to the Jordan block $\bar{F}_q \in \mathbb{R}^{n_q \times n_q}$, and $D_k \in \mathbb{R}^{(n_q-k)\times(n_q-k)}$ is a nonzero matrix in general. Observe that for all iterations $k \geq n_q$, $(\bar{\Delta}_k)_q = 0_{n_q}$. This reasoning can be applied to each Jordan block which has a zero eigenvalue. It follows that there exists an integer n_s (the size of the largest zero-eigenvalue Jordan block) such that when $k \geq n_s$,

$$\bar{\Delta}_k = \mathrm{diag}\left\{\bar{\Delta}_{nz}^k \quad 0_{q_z}\right\} \tag{7.125}$$

where q_z is the size of the whole invariant subspace corresponding to an eigenvalue of zero. With the definition $q_{nz} = n - q_z$, corresponding to the size of the whole invariant subspace corresponding to a nonzero eigenvalue, observe that $\bar{\Delta}_{nz}^k \in \mathbb{R}^{q_{nz}\times q_{nz}}$. Next observe that $\Delta_k \geq 0$ (which can be shown by applying part *2* of Lemma 7.3.1 to (7.88)), from which it follows that $\bar{\Delta}_{nz}^k \geq 0$.

Now define a lower dimensional problem by considering iterates of $\bar{\Delta}_{nz}^k$ only. Note that for $n \geq n_s$, it follows from (7.122) that $\bar{\Delta}^k_{nz}$ satisfies the following recursion:

$$\bar{\Delta}^{k+1}_{nz} = \bar{F}^T_{nz}\bar{\Delta}^k_{nz}\bar{F}_{nz} - \bar{F}^T_{nz}\bar{\Delta}^k_{nz}\bar{G}_{nz}\left(\bar{G}^T_{nz}\bar{\Delta}^k_{nz}\bar{G}_{nz} + G^T\Phi G^T + U\right)^{-1}\bar{G}^T_{nz}\bar{\Delta}^k_{nz}\bar{F}_{nz}. \quad (7.126)$$

The existence of such a recursion was alluded to in Appendix III of [4] although it was not made explicit there.

Now consider the above RDE as being associated with the factorization of the spectral matrix $\bar{\Psi}(z) = U + G^T\Phi G$ of the form (7.87) with "F" replaced by $\bar{F}_{nz}$, "G" replaced by $\bar{G}_{nz}$, "U" replaced by $U + G^T\Phi G$ and "V" replaced by zero. Factorization of this spectral matrix from the original state-space realization is trivial and the strong solution of the algebraic equation associated with (7.126) is $\bar{\Delta}_{nz} = 0$.

It can be easily checked that if (F, G) is stabilizable then $(\bar{F}_{nz}, \bar{G}_{nz})$ is also. Since we assume that $\Delta_0 \geq 0$, it follows that $\bar{\Delta}^{k=0}_{nz} \geq 0$. By construction, $\bar{F}_{nz}$ is invertible. Convergence (including rates) of $\bar{\Delta}^k_{nz}$ then follows by application of the second strengthening of Lemma 7.4.1 to $\bar{\Psi}(z)$.

Recall that the parts of the RDE iterates associated with invariant subspaces corresponding to zero eigenvalues converge in a finite number of iterations. The best and worst case convergence behaviour of $\bar{\Delta}^k_{nz}$ are therefore inherited by Δ_k, as stated in items *3 a)* and *3 b)* of Theorem 7.5.1.

7.6 Conclusions on RDE Convergence Rates.

The main purpose of the present chapter has been to establish convergence rates for a class of Riccati difference equations which arise in connection with discrete time spectral factorization. The motivation for this work in the context of this thesis is given in Chapter 6 where an algorithm for continuous time spectral factorization is developed which relies on the solution of a discrete time spectral factorization problem. The convergence rates derived in the present chapter for the RDE have immediate application in finding the convergence rate of the algorithm which is proposed in Chapter 6.

The $\frac{1}{k}$ convergence rate for the RDE which has been derived in this chapter is also of immediate relevance to certain Kalman filtering problems. In particular, if the underlying linear dynamics have unit circle modes which are uncorrupted by process noise, then these modes will give rise to invariant zeros in the realization of the discrete spectral matrix associated with the Kalman filtering problem. This problem is particularly relevant to the filtering of states associated with persistent disturbances; e.g. sinusoids and steps. The presence of unit circle invariant zeros will cause the state covariance matrix (which is governed by a Riccati difference equation) in the Kalman filter to converge to its steady state value only at a $\frac{1}{k}$ rate when it would normally converge at an exponential rate.

Appendix A

Invariant Zeros for Realizations of Nonsquare Transfer Function Matrices which are Full Rank at Infinity.

A proof of Lemma 3.1.1 in Chapter 3:

We prove only item **2** as stated in Lemma 3.1.1. Item **1** can be established by completely analogous means.

First recall that the invariant zeros of the given realization of $G(s)$ correspond to values of $\lambda \in \mathbb{C}$ where the following matrix pencil loses rank (i.e. has rank less than its normal rank):

$$\begin{pmatrix} A-\lambda I & B \\ C & D \end{pmatrix}. \tag{A.1}$$

Right multiplying $G(s)$ by the square, full rank matrix $\begin{pmatrix} D^{\dagger} & D^{\perp} \end{pmatrix}$ does not alter its invariant zeros. Thus the invariant zeros of $G(s)$ correspond also to rank loss of the matrix pencil

$$\begin{pmatrix} A-\lambda I & BD^{\dagger} & BD^{\perp} \\ C & I & 0 \end{pmatrix}. \tag{A.2}$$

Applying simple column operations preserves the rank of this matrix. Replacing the first column with itself minus the second times C results in

$$\begin{pmatrix} A-BD^{\dagger}C-\lambda I & BD^{\dagger} & BD^{\perp} \\ 0 & I & 0 \end{pmatrix} \tag{A.3}$$

which loses rank if and only if λ corresponds to an uncontrollable mode of $(A-BD^{\dagger}C, BD^{\perp})$.
□

Appendix B

$\mathcal{H}_\infty$ Riccati Equations for the ϵ-Augmented Plant.

B.1 Existence Conditions for the ϵ-Augmented Plant.

A proof of Lemma 3.3.1 in Chapter 3:

1. Suppose there exist two stabilizing solutions X_i, $i \in \{1,2\}$ of an equation of the form $A^T X + XA + XQX + R = 0$, where $A, Q, R \in \mathbb{R}^{n\times n}$ and Q, R are symmetric. Taking the difference between the equations for X_1 and X_2 yields the equation $(A+QX_2)^T(X_1 - X_2) + (X_1 - X_2)(A+QX_1) = 0$. Since, by assumption, $(A+QX_2)$ and $(A+QX_1)$ are stable, one can apply the Lemma of Lyapunov (see Lemma 0.0.1 in the summary of **Notation, Definitions and Fundamental Results** at the beginning of this thesis) to conclude that $X_1 - X_2 = 0$ and thus that stabilizing solutions are unique.

2. From Theorem 3.2.1, a γ-admissible controller for $G(s)$ exists if and only if $\exists\ \epsilon^* > 0$ such that $\forall\ \epsilon \in (0, \epsilon^*)$, there exists a γ-admissible controller for $G^\epsilon(s)$. By hypothesis $G^\epsilon(s)$is of standard form, and satisfies assumptions **A.1** ,**A.2** ,**A.3** and **A.4**. Hence Lemma 2.3.1 in Chapter 1 can be directly applied to the ϵ -augmented system to obtain existence results. The Riccati equations (3.21) and (3.22), along with the coupling condition $\rho(X_\epsilon Y_\epsilon) < \gamma^2$ are obtained by direct application of the formulae given in Lemma 2.3.1 to the realization of G^ϵ in (3.13), incorporating the formulae for $D^\epsilon_{12}, D^\epsilon_{21}$ and their inverses as given in (3.18) and (3.19) respectively. □

B.2 Equivalence of the Modified ϵ-Dependent Riccati Equations.

A proof of Lemma 3.3.3 in Chapter 3:

Since $\bar{B}_1$ and $\bar{C}_1$ are chosen according to Lemma 3.3.2, it follows (as in the proof of Lemma 3.3.1 part *1* in Appendix B.1) that nonnegative definite stabilizing solutions of equations (3.21), (3.22), (3.37) and (3.38) are unique.

$\Rightarrow$ Suppose X_ϵ and Y_ϵ are the (unique) nonnegative definite stabilizing solutions of

(3.21) and (3.22). One can use the identities (3.29) and (3.32) to eliminate the divergent terms $-\frac{1}{\epsilon^2}X_\epsilon B_2 D_{12}{}^{\perp}(D_{12}{}^{\perp})^T B_2^T X_\epsilon$ in (3.21) and $-\frac{1}{\epsilon^2}Y_\epsilon C_2^T(D_{21}{}^{\perp})^T D_{21}{}^{\perp} C_2 Y_\epsilon$ in (3.22). It follows directly that $\mathcal{X}_\epsilon = X_\epsilon$ and $\mathcal{Y}_\epsilon = Y_\epsilon$ are the (unique) nonnegative definite stabilizing solutions of (3.37) and (3.38).

$\Leftarrow$ Let $\mathcal{X}_\epsilon$ and $\mathcal{Y}_\epsilon$ be the (unique) nonnegative definite stabilizing solutions of (3.37) and (3.38). We can show by an identical argument to that in the proof of Lemma 3.3.2 that $\mathcal{X}_\epsilon$ has a structure identical with that in (3.28) and $\mathcal{Y}_\epsilon$ identical with that in (3.31). The identities (3.29) and (3.32) also hold for $\mathcal{X}_\epsilon$ and $\mathcal{Y}_\epsilon$. It follows directly that $X_\epsilon = \mathcal{X}_\epsilon$ and $Y_\epsilon = \mathcal{Y}_\epsilon$ are the (unique) nonnegative definite stabilizing solutions of (3.21) and (3.22).

Appendix C

Existence of Nonnegative Definite Stabilizing Solutions of a Family of AREs.

C.1 Existence, Differentiability and Sign Definiteness of Solutions to the Family of AREs.

A proof of Lemma 3.3.4 in Chapter 3:

Proof of item 1.
With $\eta \in \mathbb{R}$, $X \in \mathbb{R}^{n\times n}$, define the function $F : \mathbb{R} \times \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ as

$$F(\eta, X) = AX + XA^T + XQ(\eta)X. \tag{C.1}$$

By hypothesis, there is an $X_0 \geq 0$ which satisfies:

$$F(0, X_0) = 0 \tag{C.2}$$

Our next step is to establish the existence of solutions X_η to the equation

$$F(\eta, X_\eta) = 0 \tag{C.3}$$

in an interval about $\eta = 0$. We first apply the vec operation to $F(\eta, X)$:

$$\begin{aligned} \mathrm{vec}F(\eta, X) &= \mathrm{vec}(AXI) + \mathrm{vec}(IXA^T) + \mathrm{vec}(XQ(\eta)X) & \text{(C.4)} \\ &= (I \otimes A)\mathrm{vec}X + (A \otimes I)\mathrm{vec}X + (I \otimes XQ(\eta))\mathrm{vec}X & \text{(C.5)} \end{aligned}$$

Differentiation of this expression with respect to vec$X$ and application of standard identities (see [12]) results in the following expression:

$$\begin{aligned} \left.\frac{\partial \mathrm{vec}F}{\partial \mathrm{vec}X}\right|_{\{\eta=0, X=X_0\}} &= I \otimes (A + X_0Q) + (A + X_0Q) \otimes I & \text{(C.6)} \\ &= (A + X_0Q) \oplus (A + X_0Q) & \text{(C.7)} \end{aligned}$$

Here $\otimes$ and $\oplus$ denote the Kronecker product and sum respectively. It is a well known fact that the last expression has eigenvalues $\lambda_k = \lambda_{ij} = (\lambda_i(A + X_0Q) + \lambda_j(A + X_0Q))$, with $i, j \in \{1, 2, \ldots, n\}$ and thus $k = ij \in \{1, 2, \ldots, n^2\}$. Since $(A + X_0 Q)$ is stable, all eigenvalues of $(A + X_0 Q) \oplus (A + X_0 Q)$ have real parts strictly less than zero. Hence the derivative $\left.\frac{\partial \mathrm{vec} F}{\partial \mathrm{vec} X}\right|_{\{\eta=0, X=X_0\}}$ is invertible. The implicit function theorem can be immediately applied to $\mathrm{vec}F$, resulting in the conclusion that $\exists\ \bar{\eta} > 0$ and a unique continuously differentiable mapping $X_\eta \mid \mathbb{R} \to \mathbb{R}^{n \times n}$ such that $X_{\eta=0} = X_0$ and $F(\eta, X_\eta) = 0$ $\forall\ \eta \in (-\bar{\eta}, \bar{\eta})$. Thus X_η varies continuously with η in this interval and the derivative $\frac{dX_\eta}{d\eta}$ exists and is continuous.

Since X_η and $Q(\eta)$ are continuous functions of η, it is true that the real parts of the eigenvalues, $\Re\{\lambda_i(A + Q(\eta)X_\eta)\}$ also vary continuously with η. By hypothesis, when $\eta = 0$, all of these eigenvalues have negative real parts. By continuity, $\exists\ \tilde{\eta} > 0$ such that this will also be true for $\eta \in (-\tilde{\eta}, \tilde{\eta})$.

Since $F^T(\eta, X) = F(\eta, X^T)$, it follows that $X = X_\eta^T$ satisfies $F(\eta, X) = 0$ for all $\eta \in (-\bar{\eta}, \bar{\eta})$. By the uniqueness part of the implicit function theorem, it follows that $X_\eta = X_\eta^T$.

Choosing $\eta_1 = \min(\bar{\eta}, \tilde{\eta})$ completes the proof of part *1*.

Proof of item 2.
When $\frac{dX_\eta}{d\eta}$ exists, one can differentiate (3.42) to obtain the following Lyapunov equation:

$$\frac{dX_\eta}{d\eta}(A + Q(\eta)X_\eta) + (A + Q(\eta)X_\eta)^T \frac{dX_\eta}{d\eta} + X_\eta \frac{dQ}{d\eta} X_\eta = 0. \tag{C.8}$$

Let η_1 be defined as in part *1*. Let $\eta_2 = \min(\hat{\eta}, \eta_1)$, where $\hat{\eta}$ is defined in part *2* of the lemma statement. Thus when $\eta \in (0, \eta_2)$, $A + Q(\eta)X_\eta$ is a stability matrix and $\frac{dQ}{d\eta} \geq 0$. These two facts, together with (C.8), imply by the Lemma of Lyapunov (see Lemma 0.0.1 in the summary of **Notation, Definitions and Fundamental Results** at the beginning of this thesis) that $\frac{dX_\eta}{d\eta} \geq 0$, $\forall\ \eta \in (0, \eta_2)$. It follows immediately that provided $\eta \in (0, \eta_2)$, $X_\eta \geq X_0 \geq 0$. □

C.2 Existence of a Limiting Solution.

A proof of Lemma 3.3.5 in Chapter 3:

Existence of X_0.
For any $\bar{\eta} \in (0, \eta^*)$, one can apply Lemma 3.3.4 part *1* to (3.41) and conclude that X_η, the nonnegative definite stabilizing solution of (3.42) varies continuously in some interval about $\bar{\eta}$ and moreover that $\left.\frac{dX_\eta}{d\eta}\right|_{\eta=\bar{\eta}}$ exists and varies continuously in some finite interval about $\bar{\eta}$. It follows that $\frac{dX_\eta}{d\eta}$ exists and varies continuously throughout the interval $(0, \eta^*)$.

Thus one can differentiate the ARE (3.42) with respect to η to obtain (for any $\eta \in$

$(0, \eta^*)$),

$$\frac{dX_\eta}{d\eta}(A_{ZX} + Q(\eta)X_\eta) + (A_{ZX} + Q(\eta)X_\eta)^T \frac{dX_\eta}{d\eta} + X_\eta \frac{dQ(\eta)}{d\eta} X_\eta = 0$$

Note by hypothesis that that $\frac{dQ(\eta)}{d\eta} \geq 0$ and that X_η is stabilizing $\forall\ \eta \in (0, \eta^*)$, hence $A_{ZX} + Q(\eta)X_\eta$ has all eigenvalues in $\Re\{s\} < 0$. These two facts, and application of the Lemma of Lyapunov (see Lemma 0.0.1 in the summary of **Notation, Definitions and Fundamental Results** at the beginning of this thesis) to the above equation allow us to conclude that $\frac{dX_\eta}{d\eta} \geq 0$. Since X_η is monotonically increasing with η and nonnegative definite $\forall\ \eta \in (0, \eta^*)$, it must converge to some finite nonnegative symmetric matrix X_0 as $\eta \to 0$. Obviously $X_0 \leq X_\eta$. The fact that X_0 solves the ARE (3.41) follows trivially on taking the limit as $\eta \to 0$ of (3.42).

X_0 *is stabilizing.*
X_η is by hypothesis a stabilizing solution of (3.42) $\forall\ \eta \in (0, \eta^*)$. Note that X_η trivially solves $X_\eta(A + Q(\eta)X_\eta) + A^T X_\eta = 0$. Let λ be any eigenvalue (necessarily stable) of $(A + Q(\eta)X_\eta)$ with corresponding eigenvector w: $(A + Q(\eta)X_\eta)w = \lambda w$ and $\Re\{\lambda\} < 0$. Right multiplying the Riccati equation for X_η by w, we deduce that $\lambda X_\eta w + A^T X_\eta w = 0$. For this equation to hold, it is required that either $X_\eta w = 0$ or that $-\lambda$ be an eigenvalue of A. If $X_\eta w = 0$, it can be checked that that $Aw = \lambda w$ and thus that λ is also a stable eigenvalue of A. If $X_\eta w \neq 0$, then $A^T X_\eta w = -\lambda X_\eta w$ and since λ is by hypothesis stable, it must be the reflection about the origin of some unstable eigenvalue of A. Hence we deduce that

$$\lambda_i(A + Q(\eta)X_\eta) = \left\{ \begin{array}{ll} \lambda_i(A) & \text{if } \Re\{\lambda_i(A)\} < 0 \\ -\lambda_i(A) & \text{otherwise} \end{array} \right\} \tag{C.9}$$

Now since $Q(\eta)$ and X_η vary continuously with η, the eigenvalues $\lambda_i(A + Q(\eta)X_\eta)$ will also. Since they are always in the finite set $\{\pm\lambda_i(A)\}$, they are unchanged as $\eta \to 0$. Hence $\lim_{\eta\to 0}(A + Q(\eta)X_\eta) = (A + QX_0)$ has all eigenvalues in the left half-plane. □

Appendix D

Proof of the Lossless Decomposition.

Part of the proof of Lemma 4.3.3.

Observe that $LFT\{\Theta, LFT\{G_{tmp}, K\}\} = LFT\{M, K\}$ where

$$M(s) \;=\; \left(\begin{array}{cc|cc} A_\infty & B_2 D_{12}{}^{\dagger}(C_1 - E_F) & B_1 & B_2 D_{12}{}^{\dagger} D_{12} \\ -B_1 B_1^T X & \hat{A} & B_1 & B_2 \\ \hline E_F & (C_1 - E_F) & 0 & D_{12} \\ -D_{12} B_1{}^T X & \hat{C}_2 & D_{21} & 0 \end{array}\right) \tag{D.1}$$

is calculated using the state-space star product formula given in Appendix F. With the definitions of V_F, U_F and T given in equation (4.20) of Chapter 4, we apply the following state space transformation to the above realization of $M(s)$:

$$\tilde{T} = \begin{pmatrix} U_F & V_F & I \\ 0 & 0 & I \end{pmatrix}. \tag{D.2}$$

This produces an equivalent realization

$$M(s) = \left(\begin{array}{ccc|cc} \bar{A}_0 & 0 & 0 & 0 & 0 \\ \bar{A}_{01} & \bar{A}_1 & \beta_F L_F & 0 & -\beta_F (D_{12}{}^{\perp})^T \\ -B_1 B_1^T X U_F & 0 & A & B_1 & B_2 \\ \hline E_F U_F & 0 & C_1 & 0 & D_{12} \\ -D_{21} B_1^T X U_F & 0 & C_2 & D_{21} & 0 \end{array}\right) \tag{D.3}$$

where the matrices $\bar{A}_0$, $\bar{A}_{01}$ and $\bar{A}_1$ are defined according to the identity: (refer to equation (4.17) in Chapter 4)

$$T^{-1} A_X T = \begin{pmatrix} \bar{A}_0 & 0 \\ \bar{A}_{01} & \bar{A}_1 \end{pmatrix}. \tag{D.4}$$

The structure of A_X in this basis follows directly from the canonical structure introduced in in section 3.1 of Chapter 3 and from the structure of X in this basis as revealed in the proof of Lemma 4.1.2 in Chapter 4.

Since X is a stabilizing solution of the ARE (4.14) in Chapter 4 (part of the existence conditions for nonstandard controllers) , A_X is stable and hence both $\bar{A}_0$ and $\bar{A}_1$ are also stable. Observe that in the final description of $M(s)$, the modes corresponding to $\bar{A}_1$

are all unobservable and those corresponding to $\bar{A}_0$ are all uncontrollable. Elimination of these stable modes reveals that $M(s)$ is a nonminimal realization of $G(s)$, the original generalized plant.

Appendix E

The Stable Standard Output Estimation $\mathcal{H}_\infty$ Problem.

The following lemma presents the family of all $\mathcal{H}_\infty$ controllers for the standard output estimation problem which is important in the derivation of the controller structure for the nonstandard problem in Chapter 4.

Lemma E.0.1 *Let $G(s)$ be a generalized plant with the following state-space realization:*

$$G(s) = \left(\begin{array}{c|cc} A & B_1 & 0 \\ \hline C_1 & 0 & I \\ C_2 & D_{21} & 0 \end{array} \right), \tag{E.1}$$

where A is stable, D_{21} is full row rank, and $G_{21} = C_2(sI - A)^{-1}B_1 + D_{21}$ has no $j\omega$-axis zeros.

Then the following hold:

1. *An $\mathcal{H}_\infty$ controller for $G(s)$ exists if and only if the ARE*

$$\begin{aligned} 0 = {} & S(A - B_1 {D_{21}}^\dagger C_2)^T + (A - B_1 {D_{21}}^\dagger C_2)S \\ & + S[{C_1}^T C_1 - {C_2}^T ({D_{21}}^\dagger)^T {D_{21}}^\dagger C_2]S + B_1 {D_{21}}^\perp (B_1 {D_{21}}^\perp)^T \end{aligned} \tag{E.2}$$

admits a nonnegative definite stabilizing solution $S \geq 0$. By a stabilizing solution, we mean one which stabilizes $A_S = A - B_1 {D_{21}}^\dagger C_2 + S[{C_1}^T C_1 - {C_2}^T ({D_{21}}^\dagger)^T {D_{21}}^\dagger C_2]$.

Supposing that the conditions of item **1** *above are satisfied, then the following hold true:*

2. *Every $\mathcal{H}_\infty$ controller is given by $K(s) = LFT\{M, N\}$ with*

$$M(s) = \left(\begin{array}{c|cc} A + LC_2 & -L & S{C_1}^T \\ \hline -C_1 & 0 & I \\ -\Sigma C_2 & \Sigma & 0 \end{array} \right), \tag{E.3}$$

where $N(s) \in \mathcal{BH}_\infty$ is a free parameter and

$$\Sigma = (D_{21} D_{21}^T)^{-\frac{1}{2}}, \tag{E.4}$$

$$L = -B_1 {D_{21}}^\dagger - S{C_2}^T ({D_{21}}^\dagger)^T {D_{21}}^\dagger. \tag{E.5}$$

3. *The matrix $A + LC_2$ is stable and thus $M_{ij}(s) \in \mathcal{RH}_\infty$ for each partition of $M(s)$.*

4. *$M_{12}(s)$ is unimodular.*

5. *$M_{22}(s) \in \mathcal{BH}_\infty$ and $(I - M_{22}N)$ is unimodular.*

6. *$K(s) \in \mathcal{RH}_\infty$ for all $N \in \mathcal{BH}_\infty$.*

Proof:

Items **1.** and **2.** are immediate consequences of the standard state space $\mathcal{H}_\infty$ results of [31].

3. The ARE can be rewritten as

$$S(A + LC_2)^T + (A + LC_2)S + SC_2^T\Sigma^T\Sigma C_2S + SC_1^TC_1S + B_1D_{21}{}^{\perp}(B_1D_{21}{}^{\perp})^T = 0 \quad \text{(E.6)}$$

The stabilizability of $(A + LC_2, SC_1^T)$ (guaranteed by the stability of $A_S = A + LC_2 + SC_1^TC_1$), together with the fact that $S \geq 0$ ensure the stability of $A + LC_2$. This implies $M_{ij} \in \mathcal{RH}_\infty$.

4. The stability of A_S guarantees $M_{12}(s)^{-1} \in \mathcal{RH}_\infty$.

5. The stabilizing solution S of the Riccati equation, together with the bounded real lemma can be used to establish that

$$\begin{pmatrix} M_{22}^T(s) \\ V(s) \end{pmatrix} = \left(\begin{array}{c|c} (A + LC_2)^T & C_2^T\Sigma^T \\ \hline C_1S & 0 \\ B_1D_{21}{}^{\perp} & 0 \end{array}\right) \in \mathcal{BH}_\infty \quad \text{(E.7)}$$

from which it follows that $M_{22} \in \mathcal{BH}_\infty$. Since $\|M_{22}\|_\infty < 1$ and $N \in \mathcal{BH}_\infty$, we can use the small gain theorem to show that $(I - M_{22}N)^{-1} \in \mathcal{RH}_\infty$.

6. The form $K(s) = M_{11} + M_{12}N(I - M_{22}N)^{-1}M_{21}$, together with items **3** and **5** ensure the stability of $K(s)$. □

Appendix F

Star Product formula for successive LFTs.

Suppose one is given realizations of two transfer function matrices $M^i(s)$, $i = a, b$ with partitioning on the input and output signals as follows:

$$M^i(s) = \left(\begin{array}{c|cc} A^i & B_1^i & B_2^i \\ \hline C_1^i & 0 & D_{12}^i \\ C_2^i & D_{21}^i & 0 \end{array}\right). \tag{F.1}$$

Then, provided the inputs and outputs of the two systems are compatible,

$$LFT\{M^a, LFT\{M^b, K\}\} = LFT\{M^{ab}, K\}, \tag{F.2}$$

where K is an arbitrary compatible transfer function matrix and $M^{ab}(s)$ is a transfer function matrix which has the following realization:

$$M^{ab}(s) = \left(\begin{array}{cc|cc} A^a & B_2^a C_1^b & B_1^a & B_2^a D_{12}^b \\ B_1^b C_2^a & A^b & B_1^b D_{21}^a & B_2^b \\ \hline C_1^a & D_{12}^a C_1^b & 0 & D_{12}^a D_{12}^b \\ D_{21}^b C_2^a & C_2^b & D_{21}^b D_{21}^a & 0 \end{array}\right). \tag{F.3}$$

□

Appendix G

Continuous Time Spectral Matrices and The Riccati Equation.

G.1 Spectral Matrices in System and Control Theory.

a) Linear-Quadratic Regulator

The underlying system is $\dot{x} = Ax + Bu$ and the performance index is

$$\int_0^\infty [u(t)^T U u(t) + x(t)^T V x(t) + 2x(t)^T S u(t)]\, dt, \tag{G.1}$$

where S, $U = U^T$ and $V = V^T$ are real matrices. The spectral matrix is

$$\Phi(s) = U + B^T(-sI - A^T)^{-1} V (sI - A)^{-1} B + B^T(-sI - A^T)^{-1} S + S^T(sI - A)^{-1} B \tag{G.2}$$

and the Riccati equation is

$$\Pi(A - BU^{-1}S^T) + (A - BU^{-1}S^T)^T\Pi - \Pi BU^{-1}B^T\Pi + (V - SU^{-1}S^T) = 0. \tag{G.3}$$

One often assumes $S = 0$, $U > 0$ and $V = V^T \geq 0$. In this case the requirement that $\Phi(j\omega) \geq 0$ for all ω is automatic and $\Phi(s)$ is in fact nonsingular on the $j\omega$-axis.

b) Kalman Filtering Problem

The underlying system is $\dot{x} = Ax + v$, $y = Cx + w$ where $(v^T w^T)^T$ is zero mean, Gaussian white noise, with covariance

$$\mathcal{E}\left\{\begin{pmatrix} v(t) \\ w(t) \end{pmatrix} \begin{pmatrix} v(t-\tau)^T & w(t-\tau)^T \end{pmatrix}\right\} = \begin{pmatrix} V & S \\ S^T & U \end{pmatrix} \delta(\tau) \geq 0 \tag{G.4}$$

where it is assumed that $U > 0$.

The spectral matrix here is that of $y(t)$:

$$\Phi(s) = U + C(sI - A)^{-1} V (-sI - A^T)^{-1} C^T + C(sI - A)^{-1} S + S^T(-sI - A^T)^{-1} C^T. \tag{G.5}$$

The associated Riccati equation is

$$\Pi(A - SU^{-1}C)^T + (A - SU^{-1}C)\Pi - \Pi C^T U^{-1} C\Pi + V - SU^{-1}S^T = 0. \qquad (G.6)$$

c) Positive Real Lemma

The underlying system is $\dot{x} = Ax + Bu,\ y = Cx + Du$ and is positive real, implying it is stable (though not necessarily asymptotically stable), has the same number of inputs as outputs, with the requirement that

$$\Phi(j\omega) = D + D^T + C(j\omega I - A)^{-1}B + B^T(-j\omega I - A^T)^{-1}C^T \geq 0. \qquad (G.7)$$

With $D + D^T = U > 0$, the associated Riccati equation is

$$\Pi(A - BU^{-1}C) + (A - BU^{-1}C)^T\Pi - \Pi BU^{-1}B^T\Pi - C^T U^{-1} C = 0. \qquad (G.8)$$

d) Bounded Real Lemma

The underlying system is $\dot{x} = Ax + Bu,\ y = Cx$ and is bounded real, i.e., it has $\Re\{\lambda_i(A)\} < 0$ and

$$\Phi(j\omega) = I - B^T(-j\omega I - A^T)^{-1}C^T C(j\omega I - A)^{-1}B \geq 0. \qquad (G.9)$$

The associated Riccati equation is

$$\Pi A + A^T\Pi - \Pi BB^T\Pi - C^T C = 0. \qquad (G.10)$$

This is closely related to the $\mathcal{H}_\infty$ controller synthesis problem.

Remarks:
In case a) with $V = V^T \geq 0$, $S = 0$, and also when $S = 0$ in case b), $\Pi \geq 0$. Also $\Pi \leq 0$ in cases c) and d). □

G.2 Transformations of Spectral Matrices.

Notice that if $\Phi(s)$ is a spectral matrix, then so is

$$\Phi_K(s) = \left\{I + B^T(-sI - A^T)^{-1}K^T\right\}^{-1}\Phi(s)\left\{I + K(sI - A)^{-1}B\right\}^{-1}. \qquad (G.11)$$

In case a), with $A_K = A - BK$, we have

$$\begin{aligned}\Phi_K(s) &= U + B^T(-sI - A_K^T)^{-1}\{V + K^T UK - SK - K^T S^T\}(sI - A_K)^{-1}B \\ &\quad + B^T(-sI - A_K^T)^{-1}(S - K^T U) + (S - K^T U)^T(sI - A_K)^{-1}B. \qquad (G.12)\end{aligned}$$

The Riccati equation associated with $\Phi_K(s)$ in the manner that (G.3) is associated with (G.2) has the same form as (G.3). As demonstrated below, there are two consequences of these ideas, allowing us, given a stabilizability condition on (A, B), to assume (via transformation of an initially specified problem) either one of the following:

1. The new spectral matrix has $S = 0$.
2. The A matrix is stable.

To see 1) above, choose $K^T = SU^{-1}$. To see 2), we must assume that (A, B) is stabilizable. Then A_K can be taken to be stable.

Let us now restate the above claim formally.

Lemma G.2.1
Consider the Riccati equation (G.3), with (A, B) stabilizable, derived from the spectral matrix (G.2), parametrised by A, B, V, U and S.

1. *There is a second spectral matrix parametrised by A_K, B, V_K, U and $S_K = 0$ giving rise to the same Riccati equation.*
2. *There is a second spectral matrix parametrised by A_K, B, V_K, U and S_K where A_K is stable, giving rise to the same Riccati equation.*

Next we comment that one spectral matrix can have more than one Riccati equation associated with it, in the following way. Consider the spectral matrix

$$\Phi(s) = U + B^T(-sI - A^T)^{-1}V(sI - A)^{-1}B + B^T(-sI - A^T)^{-1}S + S^T(sI - A)^{-1}B \quad \text{(G.13)}$$

where (without loss of generality in light of *2* above) A is stable. Define M as the solution of $MA + A^T M = -V$ so that

$$\Phi(s) = U + B^T(-sI - A^T)^{-1}(S + MB) + (S + MB)^T(sI - A)^{-1}B. \quad \text{(G.14)}$$

Following the construction of (G.3) from (G.2), we see that the two Riccati equations associated with the two different forms of $\Phi(s)$ in (G.13) and (G.14) are

$$\begin{aligned} 0 &= \Pi_1(A - BU^{-1}S^T) + (A - BU^{-1}S^T)^T\Pi_1 - \Pi_1 BU^{-1}B^T\Pi_1 + (V - SU^{-1}S^T) \quad \text{(G.15)} \\ 0 &= \Pi_2(A - BU^{-1}S^T - BU^{-1}B^T M^T) + (A - BU^{-1}S^T - BU^{-1}B^T M^T)^T\Pi_2 \\ &\quad -\Pi_2 BU^{-1}B^T\Pi_2 - (S + MB)U^{-1}(S + MB)^T. \end{aligned}$$

It is easily verified that the solutions Π_1 and Π_2 are related by $\Pi_1 = \Pi_2 + M$. In fact, one can establish the following result.

Lemma G.2.2 *Consider the spectral matrix $\Phi(s)$ of (G.13), parametrised by A, B, V, U and S with stable A, and the associated rewriting of (G.14). Then the Riccati equation associated with (G.14) is of the type*

$$\Pi_2\hat{A} + \hat{A}^T\Pi_2 - \Pi_2 BU^{-1}B^T\Pi_2 - \hat{V} = 0 \quad \text{(G.16)}$$

with $\hat{A}$ stable and $\hat{V}$ nonnegative definite.

Remark: The point of this and the previous lemma is that all problems involving Riccati equations are more or less equivalent to one where the A matrix is stable and the V matrix is nonpositive definite, i.e., problems of the bounded-real type. Observe that one obtains the same Riccati equation by considering a spectral matrix with U replaced by I and B replaced by $BU^{-\frac{1}{2}}$. Thus one can, without loss of generality treat realizations with $U = I$. □

Proof of Lemma G.2.2: We can identify $\hat{V}$ with $(S+MB)U^{-1}(S+MB)^T$ and $\hat{A}$ with $A - BU^{-1}S^T - BU^{-1}B^TM^T$. Evidently, $\hat{V} \geq 0$. We must now consider the stability of $\hat{A}$. The equation for Π_2 can be rewritten as

$$\Pi_2 A + A^T \Pi_2 = (\Pi_2 B + S + MB)U^{-1}(\Pi_2 B + S + MB)^T \tag{G.17}$$

which shows, due to the stability of A that $\Pi_2 \leq 0$. If $(A - BU^{-1}S^T - BU^{-1}B^TM^T)x = \lambda x$ for some $x \neq 0$, then the Riccati equation for Π_2 yields

$$2\Re\{\lambda\}(x^*\Pi_2 x) = x^*\Pi_2 BU^{-1}B^T\Pi_2 x + x^*(S+MB)U^{-1}(S+MB)^T x \geq 0 \tag{G.18}$$

So if $x^*\Pi_2 x \neq 0$ and $\Re\{\lambda\} \neq 0$, it follows trivially that $\Re\{\lambda\} < 0$. If $x^*\Pi_2 x = 0$ or $\Re\{\lambda\} = 0$, then $(S+MB)^T x = 0$ and hence $Ax = \lambda x$, and $\Re\{\lambda\} < 0$ again, since A is by hypothesis asymptotically stable. So the Π_2 equation has both a stable "A" matrix, as well as a sign semi-definite constant term. □

Appendix H

Uniqueness of Strong Solutions of the Continuous Time ARE.

We assume the following: if $\Re\{\lambda\} = 0$, then $x^T A = \lambda x^T$ together with $x^T B = 0$ imply that $x = 0$.

Suppose there exist two real symmetric strong solutions P_i ($i = 1, 2$) of the Riccati equation

$$A^T P_i + P_i A + P_i R P_i + Q = 0. \tag{H.1}$$

Taking the difference between the equations for P_1 and P_2 one obtains the following:

$$\begin{aligned}(P_1 - P_2)(A + RP_1) + (A + RP_2)^T (P_1 - P_2) &= 0 & \text{(H.2)}\\ (P_1 - P_2)(A + RP_1) + (A + RP_1)^T (P_1 - P_2) &= (P_1 - P_2)R(P_1 - P_2). & \text{(H.3)}\end{aligned}$$

Without loss of generality, one may use a state-space basis for which

$$P_1 - P_2 = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \tag{H.4}$$

where $X \in \mathbb{R}^{q \times q}$ is nonsingular and $0 \leq q \leq n$.

We use this fact, along with the partitioning (conformal with that of (H.4))

$$A + RP_1 = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \tag{H.5}$$

to obtain by direct substitution in equation (H.3) the equalities

$$\begin{aligned} XA_{12} &= 0 & \text{(H.6)}\\ XA_{11} + A_{11}^T X &= \begin{pmatrix} X & 0 \end{pmatrix} R \begin{pmatrix} X \\ 0 \end{pmatrix}. & \text{(H.7)}\end{aligned}$$

Since X is by hypothesis nonsingular, the first of these equalities implies that $A_{12} = 0$. Thus $A + RP_1$ is lower block triangular in this basis:

$$A + RP_1 = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}. \tag{H.8}$$

Equation (H.7), along with nonsingularity of X, reveals that

$$A_{11}X^{-1} + X^{-1}A_{11}^T = \begin{pmatrix} I & 0 \end{pmatrix} R \begin{pmatrix} I \\ 0 \end{pmatrix}. \tag{H.9}$$

We now show that A_{11} has no imaginary axis eigenvalues. Suppose to the contrary that λ is a pure imaginary eigenvalue of A_{11} and that x_1^T is a left eigenvector corresponding to this eigenvalue. It follows directly that $\begin{pmatrix} x_1^T & 0 \end{pmatrix}$ is a left eigenvector of $A + RP_1$:

$$\begin{pmatrix} x_1^T & 0 \end{pmatrix} \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} = \lambda \begin{pmatrix} x_1^T & 0 \end{pmatrix}. \tag{H.10}$$

Multiplication of (H.9) on the left by x_1^T and on the right by $\bar{x}_1$ reveals that

$$\begin{aligned} x_1^T \begin{pmatrix} I & 0 \end{pmatrix} R \begin{pmatrix} I \\ 0 \end{pmatrix} \bar{x}_1 &= x_1^T A_{11} X^{-1} \bar{x}_1 + x_1^T X^{-1} A_{11}^T \bar{x}_1 && \text{(H.11)} \\ &= (\lambda + \bar{\lambda})(x_1^T X^{-1} \bar{x}_1) && \text{(H.12)} \\ &= 0. && \text{(H.13)} \end{aligned}$$

This, in turn, implies that

$$\begin{pmatrix} x_1^T & 0 \end{pmatrix} R = 0. \tag{H.14}$$

From this and (H.10), we have

$$\begin{pmatrix} x_1^T & 0 \end{pmatrix} A = \lambda \begin{pmatrix} x_1^T & 0 \end{pmatrix} \tag{H.15}$$

while since $R = BB^T$,

$$\begin{pmatrix} x_1^T & 0 \end{pmatrix} B = 0. \tag{H.16}$$

Together, these imply that (A, B) has an uncontrollable mode on the imaginary axis, contradicting our original hypothesis. We conclude that A_{11} has all eigenvalues in the open left half-plane.

This allows us by (H.9) to conclude that $X^{-1} \geq 0$. Identical arguments, reversing indices $1, 2$ in the above lead to $-X^{-1} \geq 0$ and thus the conclusion that $X^{-1} = 0$ and that X does not exist. Our final conclusion then is that $P_1 - P_2 = 0$ and that strong solutions are indeed unique, given our assumption on (A, B).

Appendix I

State-space Realization of Nonsingular Discrete Time Spectral Matrices.

In this appendix, we consider (generically) nonsingular spectral matrices $\tilde{\Psi}(z)$, having a realization of the form

$$\tilde{\Psi}(z) = \tilde{U}+\tilde{G}^T(z^{-1}I-\tilde{F}^T)^{-1}\tilde{S}+\tilde{S}^T(zI-\tilde{F})^{-1}\tilde{G}+\tilde{G}^T(z^{-1}I-\tilde{F}^T)^{-1}\tilde{V}(zI-\tilde{F})^{-1}\tilde{G}. \quad \text{(I.1)}$$

Unlike the continuous time case, it is possible for a nonsingular discrete spectral matrix to have a *singular* constant term $\tilde{U}$. We show in this appendix that this is no loss of generality since one can always re-express the same spectral matrix $\tilde{\Psi}(z)$ in terms of a new state-space realization for which the constant term is nonsingular. Moreover, solution of the DARE corresponding to this new realization enables a direct solution of the DARE corresponding to the original realization.

Existence of a Realization with nonsingular matrix U.

Lemma I.0.1 *Let $\tilde{\Psi}(z)$ be a nonsingular spectral matrix with a realization given in (I.1) where each constant matrix is real and $\tilde{V} = \tilde{V}^T$, $\tilde{U} = \tilde{U}^T$. Consider the family of alternative realizations of $\tilde{\Psi}(z)$ based on the set of real symmetric matrices P which is defined in the statement of Lemma 6.4.1:*

$$\tilde{\Psi}(z) = \begin{pmatrix} \hat{G}^T(z^{-1}I-\hat{F}^T)^{-1} & I \end{pmatrix} \begin{pmatrix} \hat{V}-P+\hat{F}^TP\hat{F} & \hat{S}+\hat{F}^TP\hat{G} \\ (\hat{S}+\hat{F}^TP\hat{G})^T & \hat{U}+\hat{G}^TP\hat{G} \end{pmatrix} \begin{pmatrix} (zI-\hat{F})^{-1}\hat{G} \\ I \end{pmatrix}. \quad \text{(I.2)}$$

Then the following facts hold:

1. *$\hat{\Phi}$ is a strong solution of the algebraic Riccati equation (6.51) if and only if $\bar{\Phi} = \hat{\Phi} - P$ is a strong solution of the algebraic Riccati equation associated with the new realization of $\hat{\Psi}(z)$ given in (6.42).*

2. *It is always possible to choose a matrix $P = P^*$ defining a new realization (6.42)*

of $\hat{\Psi}(z)$ such that $\bar{U} = \hat{U} + \hat{G}^T P^* \hat{G}$ is nonsingular.

Proof: Suppose $\hat{U}$ is singular and define the family of transformations

$$T = \begin{pmatrix} T_A \\ T_B \end{pmatrix} \tag{I.3}$$

where T_A is any matrix of full row rank whose row space spans the row space of $\hat{U}$ and T_B is any matrix which makes T nonsingular.

For any such T, it follows that $\hat{U}$ can be expressed thus

$$T\hat{U}T^T = \begin{pmatrix} U_A & 0 \\ 0 & 0 \end{pmatrix} \tag{I.4}$$

where U_A is a symmetric, nonsingular matrix.

Let G_A and G_B be defined according to

$$\hat{G} \begin{pmatrix} T_A^T & T_B^T \end{pmatrix} = \begin{pmatrix} G_A & G_B \end{pmatrix}. \tag{I.5}$$

It follows from the above that

$$T(\hat{U} + \hat{G}^T P \hat{G})T^T = \begin{pmatrix} U_A & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} G_A^T \\ G_B^T \end{pmatrix} P \begin{pmatrix} G_A & G_B. \end{pmatrix} \tag{I.6}$$

Moreover, G_B can be further decomposed as

$$VG_BW = \begin{pmatrix} G_3 & 0 \\ 0 & 0 \end{pmatrix} \tag{I.7}$$

where V is any matrix which achieves a row compression for G_B and W is any matrix which achieves a column compression for the matrix VG_B and hence G_3 is nonsingular.

Define the square matrix

$$Y = \begin{pmatrix} I & 0 \\ 0 & W^T \end{pmatrix} \tag{I.8}$$

which is partitioned conformally with the columns of $(G_A \;\; G_B)$ as given in (I.5).

It can now be deduced from (I.6) that

$$YT(\hat{U} + \hat{G}^T P \hat{G})T^T Y^T = \begin{pmatrix} U_A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} G_1^T & G_2^T \\ G_3^T & 0 \\ 0 & 0 \end{pmatrix} V^{-T} P V^{-1} \begin{pmatrix} G_1 & G_3 & 0 \\ G_2 & 0 & 0 \end{pmatrix}$$

where

$$VG_A = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} \tag{I.9}$$

with row partitioning conformal to that of

$$\begin{pmatrix} G_3 & 0 \\ 0 & 0 \end{pmatrix}. \tag{I.10}$$

Introduce another square matrix Z

$$Z = \begin{pmatrix} I & -G_1^T G_3^{-T} & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \tag{I.11}$$

where partitioning is in accordance with the column partitioning in (I.9) of the matrix

$$\begin{pmatrix} G_1 & G_3 & 0 \\ G_2 & 0 & 0 \end{pmatrix}. \tag{I.12}$$

With the choice

$$P^* = V^T \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} V, \tag{I.13}$$

one obtains

$$ZYT(\hat{U} + \hat{G}^T P^* \hat{G})T^T Y^T Z^T = \begin{pmatrix} U_A & 0 & 0 \\ 0 & G_3^T G_3 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{I.14}$$

Observe also that

$$ZYT \begin{pmatrix} \hat{U} & \hat{G}^T \end{pmatrix} \begin{pmatrix} T^T Y^T Z^T & 0 \\ 0 & V^T \end{pmatrix} = \begin{pmatrix} U_A & 0 & 0 & 0 & G_2^T \\ 0 & 0 & 0 & G_3^T & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{I.15}$$

From the above discussion, we deduce therefore that

$$\max_{P=P^T} \operatorname{rank}\left[\hat{G}^T P \hat{G} + \hat{U}\right] = \operatorname{rank}\left[\hat{G}^T P^* \hat{G} + \hat{U}\right] = \operatorname{rank}\left[\hat{U} \;\; \hat{G}^T\right]. \tag{I.16}$$

Note now that since $\hat{\Psi}(z)$ is nonsingular, $\begin{pmatrix} \hat{U} & \hat{G}^T \end{pmatrix}$ has full row rank. This can be shown as follows: Suppose there exists a nonzero vector v such that $v^T \begin{pmatrix} \hat{U} & \hat{G}^T \end{pmatrix} = 0$. From these facts and the realization of $\hat{\Psi}(z)$ given in (6.36), it is easy to show that $v^T \Psi(z) v \equiv 0$. However, since $\Psi(z)$ is a spectral matrix and is also generically nonsingular, there must exist values of $z = e^{j\phi}$ where $v^T \Psi(e^{j\phi}) v > 0$. This gives a contradiction which establishes the desired result.

Thus, there exists at least one matrix $P = P^*$ such that $\hat{U}$ is nonsingular. Such a P^* may be explicitly constructed by following the above proof of its existence; however, it is immediate that $\hat{U}$ will be nonsingular for almost all P. □

Suppose one is given a nonsingular spectral matrix $\tilde{\Psi}(z)$ realized according to (I.1). By choosing a matrix P^* as described in item *2* of Lemma I.0.1, one obtains a new realization given by (6.42) for which $\bar{U}$ is nonsingular.

DARE Solutions via the Transformed Spectral Matrix.

It is in the following sense that one can without loss of generality treat rational nonsingular spectral matrices which have $\tilde{U}$ nonsingular.

Suppose we know there exists a strong solution $\tilde{\Phi}$ of the ARE associated with (I.1).

Let P^* be chosen according to the above lemma. It follows from the above lemma that $\bar{\Phi} = \tilde{\bar{\Phi}} - P^*$ is a strong solution of the ARE corresponding to the new realization. By virtue of item *1* in Lemma I.0.1, $\tilde{\bar{\Phi}} = \Phi + P^*$ is the strong solution of the ARE associated with $\tilde{\Psi}(z)$. [1]

[1]The author wishes to thank David Clements for pointing out the key ideas in this proof.

Appendix J

Approximation Results for Linear Matrix Difference Equations with Unit Circle Eigenvalues.

For convenience, we review briefly some notation which is used in the proofs presented in this appendix:

1) Let $f(l)$ be a scalar valued function and $U(l)$ be a matrix valued function, both of an integer variable l. We say that $U(l) = \mathcal{O}(f(l))$ if $\sigma_{\max}(U(l)) = \mathcal{O}(f(l))$, where the notation $\mathcal{O}(\cdot)$ applied to scalar functions has the standard definition.

2) Suppose M has an even number of rows and columns, consisting of a matrix of 2×2 matrix sub-blocks; for convenience we let $[M]_{ij} \in \mathbb{R}^{2\times 2}$ denote the $(i,j)^{\text{th}}$ 2×2 subblock of M.

3) Recall the definition of the discrete time unit step function $u(\cdot)$: with l any integer, $u(l) = 1$ if $l \geq 0$ and $u(l) = 0$ if $l < 0$. *For compactness, we shall denote the discrete time unit step function with integer argument l as u_l.*

4) Given any two nonnegative integers j and k, define

$$\binom{j}{k} = \begin{cases} 0 \text{ if } j < k \\ \frac{j!}{(j-k)!k!} \text{ otherwise} \end{cases}. \tag{J.1}$$

J.1 Proof of the Approximation Formula for the Summands in $S_q(m)$.

A proof of Lemma 7.4.2 in Chapter 7:

Here we seek to understand the behaviour of the summands $A_q^l(A_q^T)^l$ in the definition

$$S_q(m) = \sum_{l=0}^{m-1} A_q^l (A_q^T)^l. \tag{J.2}$$

The formula we seek to establish here is

$$A_q^l (A_q^T)^l = C(l) + P(l) = C(l)(I + \mathcal{O}(l^{-1})), \tag{J.3}$$

where $C(l)$ and $P(l)$ are as given in the statement of Lemma 7.4.2.

With l a nonnegative integer, the following equality is fairly easily established using inductive arguments or the binomial theorem:

$$A_q^l = \begin{pmatrix} \Lambda_q^l & & & & & \\ l\Lambda_q^{l-1} & \Lambda_q^l & & & & \\ \binom{l}{2}\Lambda_q^{l-2} & l\Lambda_q^{l-1} & \Lambda_q^l & & & \\ \vdots & & \ddots & \ddots & & \\ & & & & \Lambda_q^l & \\ \binom{l}{n_q-1}\Lambda_q^{l-(n_q-1)} & & \cdots & & l\Lambda_q^{l-1} & \Lambda_q^l \end{pmatrix}. \tag{J.4}$$

Thus, A_q^l is lower block-diagonal, with the following 2×2 block entries:

$$\left[A_q^l\right]_{ij} = \begin{cases} \binom{l}{i-j}\Lambda_q^{l-(i-j)} & \text{if } i \geq j \\ 0_2 & \text{if } i < j \end{cases} \tag{J.5}$$

where 0_2 is the 2×2 zero-matrix. With the above fact in mind, observe that the following identity holds for any nonnegative integers i and j:

$$\binom{l}{i-j} = \left[\frac{l^{i-j}}{(i-j)!} + \mathcal{O}(l^{i-j-1})\right] u_{l-(i-j)}, \tag{J.6}$$

where u_n is the unit step function with integral argument n.
The following decomposition of A_q^l results from substituting (J.6) into (J.4):

$$A_q^l = B(l) + M(l) \tag{J.7}$$

where

$$B(l) = \begin{pmatrix} \Lambda_q^l & & & & & \\ l\Lambda_q^{l-1}u_{l-1} & \Lambda_q^l & & & & \\ \frac{l^2}{2}\Lambda^{l-2}u_{l-2} & l\Lambda_q^{l-1}u_{l-1} & \Lambda_q^l & & & \\ \vdots & & \ddots & \ddots & & \\ & & & & \Lambda_q^l & \\ \frac{l^{n_q-1}}{(n_q-1)!}\Lambda^{l-(n_q-1)}u_{l-(n_q-1)} & & \cdots & & l\Lambda_q^{l-1}u_{l-1} & \Lambda_q^l \end{pmatrix}. \tag{J.8}$$

Thus $B(l)$ is also lower block-diagonal, with the following 2×2 block entries:

$$[B(l)]_{ij} = \begin{cases} \frac{l^{i-j}}{(i-j)!}\Lambda_q^{l-(i-j)}u_{l-(i-j)} & \text{if } i \geq j \\ 0_2 & \text{if } i < j \end{cases} \tag{J.9}$$

It also follows from (J.6) that $M(l)$ as given in (J.7) has the following 2×2 partitions for $i > j$

$$[M(l)]_{ij} \; = \; \mathcal{O}(l^{i-j-1})\Lambda_q^{l-(i-j)}, \tag{J.10}$$

and that these matrices are zero for $i \leq j$. Also it can be easily checked that for any nonnegative integer n, the two singular values of Λ_q^n satisfy

$$\sigma_2(\Lambda_q^n) = \sigma_1(\Lambda_q^n) = 1. \tag{J.11}$$

A consequence of this is that when $i > j$,

$$\sigma_{\max}([M(l)]_{ij}) \leq \mathcal{O}(l^{i-j-1})\sigma_{\max}(\Lambda_q^{l-(i-j)} u_{l-(i-j)}) \tag{J.12}$$

and therefore

$$[M(l)]_{ij} = \mathcal{O}(l^{i-j-1}). \tag{J.13}$$

In summary then,

$$[M(l)]_{ij} = \begin{cases} \mathcal{O}(l^{i-j-1}) & \text{if } i > j \\ 0_2 & \text{if } i \leq j \end{cases}. \tag{J.14}$$

We now show that for large enough l, $B(l)$ describes the dominant behaviour of A_q^l; since when they are nonzero, each subblock $[B(l)]_{ij}$ is invertible, it follows from the above discussion that

$$\left[A_q^l\right]_{ij} = [B(l)]_{ij}\,(I_2 + \mathcal{O}(l^{-1})). \tag{J.15}$$

This equation illustrates the dominance of $B(l)$.

From the decomposition of A_q^l in (J.7), which has been derived above, it follows that

$$A_q^l (A_q^T)^l = B(l)B^T(l) + K(l) \tag{J.16}$$

where $K(l) = M(l)M^T(l) + B(l)M^T(l) + M(l)B^T(l)$. We first investigate properties of $K(l)$ and subsequently investigate $B(l)B^T(l)$. By identifying the terms of highest order in l, it can be verified from (J.14) that

$$\left[M(l)M^T(l)\right]_{ij} = \begin{cases} \mathcal{O}(l^{i+j-4}) & \text{if } i \geq 2 \text{ and } j \geq 2 \\ 0_2 & \text{if } i = 1 \text{ or } j = 1 \end{cases} \tag{J.17}$$

and from (J.14) and (J.8) it follows that

$$\left[B(l)M^T(l) + M(l)B^T(l)\right]_{ij} = \begin{cases} \mathcal{O}(l^{i+j-3}) & \text{if } i + j \geq 3 \\ 0_2 & \text{if } i = j = 1 \end{cases}. \tag{J.18}$$

From the above formulae, we conclude immediately that

$$[K(l)]_{ij} = \begin{cases} \mathcal{O}(l^{i+j-3}) & \text{if } i + j \geq 3 \\ 0_2 & \text{if } i = j = 1 \end{cases}. \tag{J.19}$$

By direct expansion of the quantity $B(l)B^T(l)$ using (J.8), followed by separation of the highest order terms into the matrix $C(l)$ for each 2×2 subblock, one obtains the identity:

$$B(l)B^T(l) = C(l) + N(l) \tag{J.20}$$

where

$$C(l) = \begin{pmatrix} \Lambda_q^l(\Lambda_q^T)^l & l\Lambda_q^l(\Lambda_q^T)^{l-1}u_{l-1} & \cdots & \frac{l^{n_q-1}}{(n_q-1)!}\Lambda_q^l(\Lambda_q^T)^{l-(n_q-1)}u_{l-(n_q-1)} \\ l\Lambda_q^{l-1}(\Lambda_q^T)^l u_{l-1} & l^2\Lambda_q^{l-1}(\Lambda_q^T)^{l-1}u_{l-1} & & \vdots \\ \vdots & & \ddots & \\ \frac{l^{n_q-1}}{(n_q-1)!}\Lambda_q^{l-(n_q-1)}(\Lambda_q^T)^l u_{l-(n_q-1)} & \cdots & \ddots & \frac{l^{2n_q-2}}{(n_q-1)!(n_q-1)!}\Lambda_q^{l-(n_q-1)}(\Lambda_q^T)^{l-(n_q-1)}u_{l-(n_q-1)} \end{pmatrix} \tag{J.21}$$

and

$$[N(l)]_{ij} = \begin{cases} \mathcal{O}(l^{i+j-4}) & \text{if } i > 1 \text{ or } j > 1 \\ 0_2 & \text{if } i = 1 \text{ or } j = 1 \end{cases}. \tag{J.22}$$

Now we consider iterates only when $l \geq (n_q - 1)$ and hence all unit step functions in (J.21) evaluate to 1. Observe also that the following identity holds for any integer $s \geq 1$:

$$\Lambda_q^s(\Lambda_q^T)^s = (|\sigma_q|^2 + |\omega_q|^2)^s I_2 \tag{J.23}$$

where $|\sigma_q|$ and $|\omega_q|$ are the magnitudes of the real and imaginary parts of the eigenvalues of A_q. In particular, for a unit circle eigenvalue, the right-hand side is simply I_2. When applied to (J.21), this result leads to the equivalent form for $C(l)$ which is presented in the lemma statement. It follows from (J.22) together with the invertibility of each $[C(l)]_{ij}$ and the identity (J.20) that when $l \geq (n_q - 1)$,

$$\left[B(l)B^T(l)\right]_{ij} = [C(l)]_{ij}\,(I_2 + \mathcal{O}(l^{-2})) \tag{J.24}$$

and it is in this sense that $C(l)$ is best seen to approximate $B(l)B^T(l)$.

The first equality in equation (7.46) of Lemma 7.4.2 can now be obtained:

$$A_q^l(A_q^T)^l = C(l) + P(l), \tag{J.25}$$

with the definition $P(l) = K(l) + N(l)$. Note that the order of $P(l)$ is inherited from the matrix $K(l)$,

$$[P(l)]_{ij} = \begin{cases} \mathcal{O}(l^{i+j-3}) & \text{if } i + j \geq 3 \\ 0_2 & \text{if } i = j = 1 \end{cases}. \tag{J.26}$$

It follows from (J.26), together with the invertibility of each $[C(l)]_{ij}$, and the equality $A_q^l(A_q^T)^l = C(l) + P(l)$ that when $l \geq (n_q - 1)$,

$$\left[A_q^l(A_q^T)^l\right]_{ij} = [C(l)]_{ij}\,(I_2 + \mathcal{O}(l^{-1})) \tag{J.27}$$

and it is in this sense that $C(l)$ is best seen to approximate $A^l(A^T)^l$. The following equality is a consequence of (J.27)

$$A_q^l(A_q^T)^l = C(l)(I + \mathcal{O}(l^{-1})). \tag{J.28}$$

□

J.2 Proof of the Approximation Formula for $S_q(m)$.

A proof of Lemma 7.4.3 in Chapter 7:

Recall the definition of $S_q(m)$ for $m \geq 1$

$$S_q(m) = \sum_{l=0}^{m-1} A_q^l (A_q^T)^l. \tag{J.29}$$

Observe that it follows immediately from Lemma 7.4.2 that

$$S_q(m) = \sum_{l=0}^{m-1} C(l) + \sum_{l=0}^{m-1} P(l). \tag{J.30}$$

Consider the 2×2 subblocks of the first sum in (J.30) for $i \geq j$:

$$\left[\sum_{l=0}^{m-1} C(l)\right]_{ij} = \sum_{l=0}^{m-1} \frac{l^{i+j-2}}{(i-1)!(j-1)!} (\Lambda_q^T)^{i-j} u_{l-(i-1)}. \tag{J.31}$$

When $m \geq (n_q - 1)$, each of the 2×2 subblocks of the first sum for $i \geq j$ can be written as follows:

$$\left[\sum_{l=0}^{m-1} C(l)\right]_{ij} = \frac{1}{(i-1)!(j-1)!} (\Lambda_q^T)^{i-j} \sum_{l=0}^{m-1} l^{i+j-2} - E_{ij} \tag{J.32}$$

where E_{ij} is a constant matrix which has the values $E_{ij} = \sum_{l=0}^{i-2} \frac{l^{i+j-2}}{(i-1)!(j-1)!} (\Lambda_q^T)^{i-j}$ when $i \geq 2$ and $E_{ij} = 0$ when $i = 1$.

Given any integer r, the following identity is a standard result which can be obtained by approximating $\sum_{l=0}^{m-1} l^r$ by the definite integral $\int_0^{m-1} x^r dx$:

$$\sum_{l=0}^{m-1} l^r = \frac{m^{(r+1)}}{(r+1)} + O(m^r). \tag{J.33}$$

When the above observation is applied to (J.32), the following simplification results:

$$\sum_{l=1}^{m-1} C(l) = D(m) + G_1(m), \tag{J.34}$$

where

$$D(m) = \begin{pmatrix} mI_2 & \frac{m^2}{2}\Lambda_q & \frac{m^3}{6}\Lambda_q^2 & \cdots & \frac{m^{n_q}}{(n_q-1)!n_q}\Lambda^{(n_q-1)} \\ \frac{m^2}{2}(\Lambda_q^T) & \frac{m^3}{3}I_2 & \frac{m^4}{8}\Lambda_q & & \vdots \\ \frac{m^3}{6}(\Lambda_q^T)^2 & \frac{m^4}{8}(\Lambda_q^T) & \frac{m^5}{20}I_2 & & \\ \vdots & & & \ddots & \\ \frac{m^{n_q}}{(n_q-1)!n_q}(\Lambda_q^T)^{n_q-1} & \cdots & & & \frac{m^{2n_q-1}}{(n_q-1)!(n_q-1)!(2n_q-1)} I_2 \end{pmatrix},$$

which can also be expressed as

$$[D(m)]_{ij} = (\Lambda_q^T)^{i-j}\frac{m^{i+j-1}}{(i-1)!(j-1)!(i+j-1)}, \tag{J.35}$$

and where

$$[G_1(m)]_{ij} = \mathcal{O}(m^{i+j-2}) - E_{ij} = \mathcal{O}(m^{i+j-2}). \tag{J.36}$$

Now examine the second summation in the expression (J.30):

$$[G_2(m)]_{ij} = \sum_{l=0}^{m-1} [P(m)]_{ij} = \sum_{l=0}^{m-1} \mathcal{O}(l^{i+j-3}), \tag{J.37}$$

from which it follows that

$$[G_2(m)]_{ij} = \begin{cases} \mathcal{O}(m^{i+j-2}) & \text{if } i+j \geq 3 \\ 0_2 & \text{if } i=j=1 \end{cases} . \tag{J.38}$$

Defining $G(m) = G_1(m) + G_2(m)$, one obtains

$$S_q(m) = D(m) + G(m), \tag{J.39}$$

where $G(m)$ has the order given in the lemma statement. From the equality (J.39) together with the invertibility of $[D(m)]_{ij}$, it follows that

$$[S_q(m)]_{ij} = [D(m)]_{ij}\,(I_2 + \mathcal{O}(m^{-1})) \tag{J.40}$$

from which the identity $S_q(m) = D(m)(I + \mathcal{O}(m^{-1}))$ follows.

It can now be verified that $D(m)$ has the form stated in equation (7.53) in Lemma 7.4.3 where the matrix Θ in that equation is defined as follows:

$$\Theta = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{6} & \cdots & \frac{1}{(n_q-1)!n_q} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{8} & & \vdots \\ \frac{1}{6} & \frac{1}{8} & \frac{1}{20} & & \\ \vdots & & & \ddots & \\ \frac{1}{(n_q-1)!n_q} & \cdots & & & \frac{1}{(n_q-1)!(n_q-1)!(2n_q-1)} \end{pmatrix} \otimes I_2 \tag{J.41}$$

$$[\Theta]_{ij} = \frac{1}{(i-1)!(j-1)!(i+j-1)} I_2. \tag{J.42}$$

The fact that $\Theta > 0$ follows from the following discussion:
First define

$$\psi(x) = \begin{pmatrix} 1 \\ x \\ \frac{x^2}{2} \\ \vdots \\ \frac{x^{n_q-1}}{(n_q-1)!} \end{pmatrix} \otimes I_2 \tag{J.43}$$

and then observe that

$$\Theta = \int_0^1 \psi(x)\psi^T(x)\,dx \tag{J.44}$$

from which it follows immediately that $\Theta \geq 0$. We show now that in fact $\Theta > 0$: suppose there exists a vector $\alpha \neq 0$ such that

$$\Theta\alpha = 0. \tag{J.45}$$

Then

$$\alpha^T \Theta \alpha = 0 \tag{J.46}$$

which, due to the identity (J.44) can be alternatively expressed as

$$\int_0^1 \alpha^T \psi(x) \psi^T(x) \alpha \, dx = 0. \tag{J.47}$$

An immediate consequence of this is that $\alpha^T \psi(x) = (0 \quad 0)$, for all $x \in [0,1]$. Let $\alpha^T = \left(\alpha_1^T \quad \dots \quad \alpha_{n_q}^T\right)$ with $\alpha_r^T = (\gamma_r \quad \beta_r)$ where γ_r and β_r are the real entries of the vectors $\alpha_r \in \mathbb{R}^{2\times 2}$, $r = 1, 2, \dots, n_q$. Thus, for all $x \in [0,1]$ it follows that

$$\alpha^T \psi(x) = \sum_{r=1}^{n_q} \alpha_r^T \frac{x^{r-1}}{(r-1)!} I_2 = (0 \quad 0) \tag{J.48}$$

and therefore that

$$\sum_{r=1}^{n_q} \gamma_r \frac{x^{r-1}}{(r-1)!} \;=\; 0, \tag{J.49}$$

$$\sum_{r=1}^{n_q} \beta_r \frac{x^{r-1}}{(r-1)!} \;=\; 0, \tag{J.50}$$

$$\tag{J.51}$$

also for all $x \in [0,1]$. This implies that γ_r and β_r are zero for all r which implies that $\alpha = 0$ which contradicts our earlier assumption that $\alpha \neq 0$. Hence $\Theta > 0$. □

Appendix K

Convergence Rate of a Time-varying Linear Matrix Difference Equation.

A proof of Lemma 7.5.1 in Chapter 7:

Recall the linear matrix difference equation which was introduced in Lemma 7.5.1 of Chapter 7:

$$\Xi_{k+1} = A_k \Xi_k B_k + \Upsilon_k. \tag{K.1}$$

With the definitions $x_k = \operatorname{vec} \Xi_k$, $\mathcal{A}_k = (B_k^T \otimes A_k)$ and $v_k = \operatorname{vec}\Upsilon_k$, application of the vec operation to (K.1) yields the vector difference equation:

$$x_{k+1} = \mathcal{A}_k x_k + v_k. \tag{K.2}$$

Note in particular that $\mathcal{A}_k \to \mathcal{A}$ where $\mathcal{A} = (B^T \otimes A)$ is a stable matrix since all the eigenvalues of $\mathcal{A}$ can be given as $\lambda_{ij}(\mathcal{A}) = \lambda_i(A)\lambda_j(B)$. We now consider (K.2) in the light of the well established stability theory for difference equations (see for example [52]). We do so via a series of lemmas which follow. The convergence rates established for the (vector) difference equation (K.2) can then be used to infer the convergence rates for the matrix difference equation (K.1).

The following result is well known, although a statement could not be readily found in the literature.

Lemma K.0.1 *Suppose one is given a bounded sequence of matrices $\{\mathcal{A}_k\}$ which has a limit $\mathcal{A} = \lim_{k\to\infty} \mathcal{A}_k$ with each of its eigenvalues satisfying $|\lambda_i(\mathcal{A})| < 1$. Then the homogeneous linear difference equation*

$$z_{k+1} = \mathcal{A}_k z_k \tag{K.3}$$

is exponentially asymptotically stable.

Before presenting a proof of this result, we recount the following well-known stability result for difference equations (see Theorem 4.7.2 in [52]).

Lemma K.0.2 *Suppose one is given a sequence of matrices $\{F_k\}$ such that the associated linear homogeneous difference equation*

$$\tilde{y}_{k+1} = F_k \tilde{y}_k \tag{K.4}$$

is uniformly asymptotically stable. Now consider the nonhomogeneous difference equation

$$y_{k+1} = F_k y_k + f(k, y_k). \tag{K.5}$$

If $f(k, y_k)$ is such that $\|f(k, y_k)\| \leq L\|y_k\|$ for sufficiently small L, then the solution y_k of (K.5) is exponentially asymptotically stable.

Proof of Lemma K.0.1 : The continuous time analogue of Lemma K.0.1 is well known and widely stated (see e.g. section 2.3.2 in [83]). The discrete time result is well known, but seems not to be widely stated. We now present a brief proof of the stated stability result for (K.3) based on Lemma K.0.2. We shall apply Lemma K.0.2 to the difference equation (K.3) by making the following associations in (K.5): $y_k = z_k$, $F_k = \mathcal{A}$ (for all k) and $f(k, y_k) = (\mathcal{A}_k - \mathcal{A})y_k = \delta_k y_k$ where $\delta_k = \mathcal{A}_k - \mathcal{A}$. By hypothesis, for any $\epsilon > 0$, there exists an integer $N(\epsilon)$ such that $k \geq N(\epsilon)$ implies that $\|\delta_k\| \leq \epsilon$. Observe therefore that for any $\epsilon > 0$, there exists some $N(\epsilon)$ such that $k \geq N(\epsilon)$ implies $\|f(k, y_k)\| \leq \|\delta_k\|\|y_k\| \leq \epsilon\|y_k\|$. Thus the condition on L in Lemma K.0.2 can be satisfied by making k large enough and we can conclude that (K.3) is exponentially asymptotically stable. □

Lemma K.0.3 *Suppose the homogeneous linear difference equation (K.3) is exponentially asymptotically stable. If in the nonhomogeneous equation (K.2) $v_k = \mathcal{O}(\frac{1}{k})$ then $x_k = \mathcal{O}(\frac{1}{k})$.*

Proof: The author is not aware of a statement or proof of the discrete time result in the literature. A similar but not directly analogous result for continuous time systems is established in Lemma 2.6 of [83].

Let $\Gamma(k, k_0)$ be the transition matrix associated with the homogeneous difference equation (K.3). Since it is asymptotically stable, there exist constants $a > 0$ and $0 < \eta < 1$ such that $\|\Gamma(k, k_0)\| \leq a\eta^{k-k_0}$. Now we apply to (K.2) the well-known variation of constants formula (see e.g. section 4.6 of [52]) to obtain $x_k = \Gamma(k, 0)x_0 + \sum_{j=0}^{k-1} \Gamma(k, j+1)v_j$. Since $v_k = \mathcal{O}(\frac{1}{k})$, there exists a constant $b > 0$ such that one can write $\|v_j\| \leq \frac{b}{j+1}$ for all $j \geq 1$. By taking the norm of the variation of constants formula for x_k and by repeated application of the triangle inequality, one obtains $\|x_k\| \leq a\eta^k\|x_0\| + ab\sigma_k$ where $\sigma_0 = 0$ and $\sigma_k = \sum_{l=1}^{k} \frac{\eta^{k-l}}{l}$ when $k \geq 1$. Note that σ_k satisfies the scalar difference equation $\sigma_{k+1} = \eta\sigma_k + \frac{1}{1+k}$. We now show that the solution of this equation satisfies $\sigma_k = \mathcal{O}(\frac{1}{k})$ which establishes the desired result.

First define $\omega_k = k\sigma_k$ for all $k \geq 1$ and note that it follows directly from the recursion for σ_k that $\omega_0 = 0$ and $\omega_{k+1} = (1 + \frac{1}{k})\eta\omega_k + 1$. This equation has a steady-state solution

$\omega = \frac{1}{1-\eta}$. The difference equation for ω_k can be rewritten in terms of $\delta\omega_k = \omega_k - \omega$;

$$\delta\omega_{k+1} = \eta(1 + \frac{1}{k})\delta\omega_k + \frac{1}{k}\eta\omega \tag{K.6}$$

Application of Lemma K.0.1 establishes the exponential asymptotic stability of the homogeneous equation associated with (K.6) (since $\lim_{k\to\infty} = \eta$ and $|\eta| < 1$). Since $\frac{1}{k}\eta\omega \to 0$, it follows from (K.6) that $\delta\omega_k \to 0$.

In summary, $\omega_k \to \omega > 0$ and therefore for all $\epsilon > 0$ there exists a $N(\epsilon)$ such that $k \geq N(\epsilon)$ implies that $|\omega_k - \omega| \leq \epsilon$. Therefore if $k \geq N(\epsilon)$, $|\sigma_k - \frac{\omega}{k}| \leq \frac{\epsilon}{k}$ and hence $\sigma_k \leq \frac{\omega+\epsilon}{k}$, which establishes that $\sigma_k = \mathcal{O}(\frac{1}{k})$. $\square$

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